

John Lodge Cowley's
GEOMETRY made Easy!

from O. C. Jacquet

A new and methodical Explanation of
the ELEMENTS of GEOMETRY.

CONTAINING

EUCLID'S ELEMENTS, and the most material Propositions of *Archimedes*, concerning the *Cylinder*, *Cone* and *Sphere*. A concise Treatise of *Algebra*, and its Application to *Geometry*. And an Introduction to *Conic-Sections*. From the *French*, with several Additions, and Amendments.

By JOHN LODGE COWLEY,
Teacher of MATHEMATICS.

By whom is added,

A curious and exact Method of representing the various Kinds of Solids, and their Sections, by new invented Schemes, which are printed upon Paste Board, and so constructed, that when cut out and duly folded together; they distinctly shew each Section; and at the same time constitute the very Solid itself, in its real and natural Form; whereby the Doctrine of Solids will be much easier comprehended, than by any other Method yet published.

The whole,

Delivered in a most easy and concise Manner, is chiefly calculated for the Use of MATHEMATICAL SEMINARIES, and such as would be acquainted with the Principles of this Science, by their own Application only.

L O N D O N:

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P R E F A C E.

THE Excellency and Usefulness of the Science, which is the Subject of the following Sheets, have been for many Ages too well known and experienced to need any Encomium at this Time of Day; and indeed an Attempt to shew the great Advantages, which Mankind hath reap'd from the Knowledge of the Mathematics, were as ridiculous as to endeavour the exact Quadrature of the Circle, or the investigating the true Root of a Surd Number. I shall therefore only beg leave to acquaint the Reader, that my principal View in sending abroad this Work, which contains the first Rudiments of Geometry, &c. was to introduce him to the Acquaintance of those learned Authors who have entered more deeply into this Science; and to render it fit for the Use of Mathematical Seminaries, or of such as would attain to the Knowledge of Geometry by their own Application. Perspicuity and Brevity are therefore the Ends here chiefly aim'd at; in order to which, I have retrench'd or added where I found my Author too superfluous, or obscurely concise.

And for the better understanding the various Kinds of Solids, which are so perplexing to young Minds, and frequently deter 'em from the Prosecution of these Studies, I have added an entire new Method of representing them and their several Sections on Paste Board, so as to exhibit at one View their different Forms and Proportions; by which Means the Difficulties which have hitherto attended the Contemplation of those Bodies, when expressed by Lines, are entirely obviated, and the Doctrine of Solids render'd perfectly plain and easy,

without any previous Knowledge in the Principles of Perspective.

Altho' it be not the Province of this Treatise, as a Commentary on the Elements of Euclid, to treat of more than the Five regular Solids, yet as those Bodies may be transformed into many irregular ones; I thought it might not be a useless Curiosity, to present the Reader with a Delineation of the Plat forms of some few of them, according to this new Method.

Neither the Price nor the Bounds which I prescribed to this Book when I first publish'd my Proposals for a Subscription, would permit me to swell it with any farther Additions; but if what is here offer'd for explaining the Doctrine of Solids contain'd in Euclid's XI and XII Books, be approv'd, I shall hereafter, as I have Leisure and Opportunity, extend it to his other Propositions,

It may perhaps be ask'd what should induce me to increase the Number of Books which are already Extant on this Subject, and more especially as some very eminent Mathematicians have so warmly inveigh'd against all Performances of this Nature, and asserted Euclid's Method of ranging and demonstrating the Propositions to be better than any other? &c. &c. To this I shall only answer, that in my own Experience the Knowledge of Geometry hath by the following Method been easily attain'd by those to whom Euclid's prov'd very difficult; and this in the Judgment of the impartial Reader will, no doubt, sufficiently justify me in this Attempt.

London, July 23,

1752.

J. L. C.

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A T A B L E

Shewing how to find the Propositions of Euclid treated of in this Work.

The first Figure denotes the Proposition of Euclid ; and the second, which is a Roman Figure, alludes to the Book ; and the third, in what Number thereof, it is contained in this Treatise.

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Errata.

E R R A T A.

THE marginal Numbers from 28 in Page 8, to 52 in p. 13, should each be accounted an Unit more, 28 should be 29, and so on. P. 13, l. 7, f. EA, r. ED. P. 16, numb. 71, l. 5, f. CD upon D, r. CD upon Z. P. 20, l. 9, f. D, r. B. P. 21, l. 18, f. BD, r. CD. P. 23, l. 9, make a full-stop at BD. P. 24, l. 3, leave out, as CD. FG. &c. P. 30, numb. 21, l. 8, f. EAD, r. EAB. P. 31, l. 4, f. BE, r. BF. *Ibid.* l. 20, f. ED, r. BD. *Ib.* l. 35, f. GBF, r. GEF. P. 42, l. 11, f. an Isocles, r. two Hoctes, and erase the Word of, in the l. above. P. 48, l. 9, f. DEF, r. DFE. *Ib.* num. 116, l. 5, f. BC, r. AB. P. 61, l. 33, f. ABCD×E, r. ABCD×DE. P. 63, rule 2, l. 7, remove the Word by, and put it next the Word much. P. 64, num. 11, l. 8, next to the Word makes, read as follows; only 6, whereas 3 times 3, or 3 multiplied by itself, makes. P. 67, Prop. 6, l. 5, put the Word half, next the Word of. *Ib.* l. 14, f. BC, r. BC. P. 68, l. 10, r. 2 AB×BC=2bd+2dd. next l. r. 2AB×BC, &c.

-2 -2

next l. r. AB+BC. *Ib.* l. 24, after the Word adding, r. AC to one side and its equal bb to the other. P. 86, l. 34, f. A+C, r. A+D. P. 89, l. 20, f. the part B. r. BH. P. 92, f. E and F, r. C and F. P. 103, num. 51, l. 5, f. CEFK, r. CERH. P. 105, l. 1, f. CE, r. CD. P. 106, l. the last, for BD×AE ×BD, r. BD×AE+BD. P. 108, l. 8 f. BG, r. EG. P. 113, l. 28 f. GAI, r. GHI. *Ib.* l. 30, f. ABCD, r. ABCE. P. 143, l. 4, f. ED, r. BD. *Ib.* l. 7, f. AB, r. AP. P. 155, l. the last but three, f. GM, r. CM. P. 156, l. 9, f. BG, r. BO. P. 162, l. 38, f. ADEGLE, r. ADEGLF. P. 164, last l. but two, f. these two numbers, r. three numbers. P. 170, l. 18 near its end, f. nn×2ss, r. nn=2ss. P. 174, l. 14, f. Y, r. X. *Ib.* l. 3, f. ICF, r. IGF. P. 178, l. 27, f. X, r. +. *Ib.* l. the last, f. X, r. +. P. 181, l. 42, f. EH, r. FH. P. 188, last line but three, next to GH, r. by FG. P. 203, l. 38, f. x-, r. x-b. P. 205, l. the last, f. ac+d, r. ac+dx. P. 206, l. 15, f. c-, r.

-2 -2

c-a. P. 212, l. 11, f. EP, r. FP. P. 213, l. 18, f. EMP, r. FMP.

A N
E X P L A N A T I O N
O F T H E
T E R M S a n d C H A R A C T E R S
U s e d i n t h i s T R E A T I S E.

A X I O M.

Is a self evident and immutable Truth, perceived and consented to by every one at first sight.

D E M A N D.

Is a Proposition which doth not appear Self-evident at first sight, but becomes so soon as it is considered, and is therefore justly required to be granted as undeniable.

D E F I N I T I O N.

Is a Proposition that explains the meaning of a Word; or which gives an exact Idea of the thing Signified by that Word.

T H E O R E M.

Is likewise called a Proposition, the truth of which must be Demonstrated.

P R O B L E M.

Is also a Proposition which must be demonstrated, but there is in it something required to be done; and that which was required to do, must be proved that it is done.

L E M M A.

Is a Proposition made use of, in order to abbreviate the Proof of the following Demonstration.

C O R A L L A R Y.

Is a truth or consequent Proposition, gained from some preceding Demonstration.

B

D E M O N.

DEMONSTRATION.

Is a train of arguments, deduced from such plain Axioms and other Self-evident Truths, as cannot be denied by any One that considers them, but convince the Mind of the truth thereof, even of the most obstinate, who will grant nothing but what is extorted by its undeniableness, and is the highest degree of Proof, human Reason is capable of attaining to.

SCHOLIUM.

Is a brief commentary, or Observation upon some precedent Discourse.

N. B. EUCLID includes the Terms *Theorem* and *Problem* both in one general Term *Proposition*.

CHARACTERS.

This mark $+$ signifies *plus*, or *more*, $A+B$, that is A more B, This—Signifies *Minus* or *less*, $A-B$, Signifies A less B. $=$ is the Mark or Sign of *Equality*. $C=D$, signifies that C is *equal* to D.

$>$ Greater, $<$ Lesser.

X By, This is the Sign of Multiplication, as AXD signifies A multiplied by D.

Sup. *Supra* or *above*. b. *Book*. n. *Number*.

There are figures put in the Margin of this Book, which shew how to find the Propositions alluded to, thus b. 2. n. 6. signifies *Book the Second, Number the Sixth*. But if the Place referred to, is in the same Book, then it is cited with this note *sup.* thus *sup. n. 5.*—that is *number the fifth above*.

When the Proposition treated of is to be found in Euclid the Place is denoted thus: *Eucl. 1. Prop. 7.* that is, *Euclid Book the First, Proposition the Seventh*.

As to what other Notes may be used in this Work, they will be explained in the Places where they first occur; and that the Use of the above Characters may become Familiar, I shall make use of them in proposing the following Truths.

AXIOMS.

1. THE whole is greater than its part.

If A and B are the parts of the Line X, the whole Line X is greater than either A or B taken separately.

2. THE Whole is equal to all its parts taken together.

If A and B are all the parts of X, it is evident that $A+B$, that is A with B is equal to X: which is expressed thus, $A+B=X$.

3. QUAN-

3. QUANTITIES which are Equal to the same Quantity are equal one to another.

Suppose $A=Z$ and $B=Z$; that is A equal to Z and B also equal to Z, then A and B are two equal Quantities, this reasoning is expressed thus: if $A=Z$ and $B=Z$; or (which is the same) if $A=Z=B$: then $A=B$. I shall often use this Expression: it must therefore be well observed. To this AXIOM may be added the following one, which is no less Evident: If A is equal to B, every quantity greater or less than B, is likewise greater or less than A.

4. IF equal Quantities are added to equal Ones, their Sums will be Equal.

If $A=B$, the same Quantity X being added to A and B, they will continue equal $A+X=B+X$.

5. IF from equal Quantities equal ones be Subtracted, the Remainders are equal.

If $A=B$, then $A-X=B-X$; that is if A and B are two equal Quantities, then A less X is equal to B less X.

6. IF equal Quantities be added to unequal Ones, they will continue unequal, greatest that which was greater, and lesser that which was lesser.

If X and Z are two unequal Quantities, and A and B two equal Ones, $X+A$ and $Z+B$ are unequal, that greater or lesser than the Other, according as they were before.

7. IF equal Quantities be subtracted from unequal ones, the Remainders will be unequal.

That is, if X and Z are unequal Quantities, $X-A$ and $Z-A$ are unequal, one greater or lesser than the other, according as X and Z were before.

8. A Quantity having the Sign+, being added to itself or its Equal having the Sign—, is equal to nothing, or Cypher, That is, $+A-A=0$.

9. THINGS which are the half or third &c. of the same or equal Quantities, are Equal, unequal if the whole Quantities, are unequal, and Greater or Lesser, according as the Wholes are greater or lesser.

10. QUANTITIES which every way coincide together when placed one upon the other, are equal.

If two Lines placed one upon the other do exactly coincide, they are equal.

A D V E R T I S E M E N T

To these Axioms may be added this Proposition, that a thing is true by Construction, when it is done exactly according to the Rule which was agreed upon. Thus if it be agreed upon to divide a Line into two equal Parts and that AC be so divided at the point B , then by Construction AB and BC are each, the half of that Line AC .



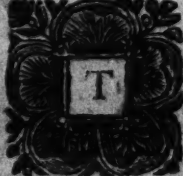
A Proposition is also incontestable when it cannot be denied without advancing an Absurdity, or what is manifestly false.

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E L E M E N T S
OF
G E O M E T R Y,
BOOK I. SECTION I.

OF THE
Different Species of EXTENSION,
D E F I N I T I O N S.

I.  HERE are three Sorts of Extension, Length, Breadth, and Depth, which are the three Dimensions of Body.

The Object of Geometry is not that material Extension of such Bodies as are really extended into Length, Breadth, and Depth, it is such an intelligible Extension, as the Mind conceives, so that were there no Bodies in the World, yet what Geometers demonstrate of Extension would nevertheless be true; hence it is, that altho' Bodies are liable to alterations yet the Truths of Geometry remain immutable; for they do not depend upon Matter, but upon clear and distinct Ideas which are in the Mind, therefore altho' there is no Body but what hath three Dimensions, yet one may be considered without the other, Length may be considered without Breadth, and Breadth without Depth, as the length of a Road may be considered without reflecting on its Breadth, and the Breadth without thinking on the Depth of the Earth; the Idea of Length excludes those of Breadth and Depth, and that of Breadth excludes that of Depth, and these Notions are true, altho' in reality those three things are inseparable, because in the Manner they are conceived, they are distinguished in this, that the one is considered without the other, therefore Geometricians can in this manner suppose things to be Long without being Broad, and Broad without being Deep, or Thick. Should any one

one assert that these Suppositions are utterly false, yet the Consequences which are deduced from them, are not to be rejected as Erroneous. For Example, Altho' there is not in fact an exact Circle in the World, yet according to the Supposition of a Circle being a Figure whose Circumference is every where equally distant from its Center, it must needs follow, that all lines drawn from its Center to its Circumference are equal.

2 II. *A Point is that which hath no part.*

That is, the parts thereof are not considered; for every thing which is extended hath parts: Now if the Point be something, it is extended, therefore it is said to be without parts because what it hath are not regarded, it being considered as indivisible.

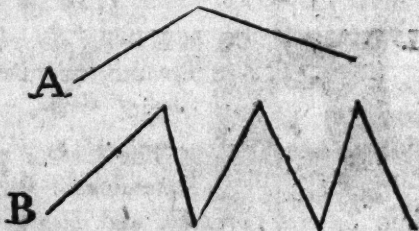
3 III. *A Line is Length without Breadth.*

That is, an Extension which is supposed either to have no breadth or whose breadth is not considered, it may also be conceived to be the Trace or Fluxion made by the Motion or Change of Place of a Point. There are two sorts of Lines, the right Line, and curve Line.

4 IV. *The right Line is the Shortest which can be drawn between two given points.*

5 V. *CURVE LINES are those which are not the Shortest between two given points.*

There are many Sorts of curve Lines, regular and irregular, Geometrical and Mechanical, but these we shall not treat of here. Lines which are compounded of two or more Lines as A and B are called crooked, the Line A composed of two Lines, and the Line B made of several Lines, may either of them be considered as only one Line, therefore to distinguish them they must be called crooked.



6 VI. *A Surface is an Extension long and broad without Depth.*

7 VII. *A right or plane Surface, is the Shortest between two right Lines.*

A right Surface may be conceived to be generated by the Motion of a right Line.

8 VIII. *A Curve Surface is greater than the right or plane Surface; which is between the same two Lines.*

9 IX. *A Solid is an Extension which hath three Dimensions, Length, Breadth, and Depth.*

A Solid is generated by the flux or motion of a Surface. Every Solid is a Body, as the very Name imports, for that is Extension which hath Length, Breadth and Depth, therefore, Extension and Body is the same Thing.

SECT.

S E C T. II.

Of Length which is the first and simplest Dimension of Body.
Of right Lines.

EVIDENT Propositions concerning right Lines, or Corollaries proceeding
from the Definition thereof.

PROPOSITIONS.

I. **T**HE Ends or Extremities of a Line are two Points. 10
II. WHEN two different Lines intersect, their Section is an
indivisible Point. 11

III. IF a Line drawn between two Points, is farther from one Part 12
than another of a right Line drawn between the same two Points, it is
longer than that right Line. Plate 1. Fig. 1.

The Curve Lines ACB and ADB are longer than AB, and the
crooked Lines EFG and BHG are longer than EG, this follows
from the true Idea of a right Line, which is the shortest that can be
drawn between the same two Points.

IV. Two Points being given, to draw a right Line from one to the 13
other.

This is easily done by a Rule, or a Thread stretched from one
Point to the other.

V. A right Line given may be prolonged. 14

This may also be done by help of a Rule. But when the given
Line is very short, it will be difficult to place the Rule so that the
produced Part may be in a direct Line with the given one, I
shall therefore shew how to do it geometrically, when we come to
the *Recreative Problems* at the End of Book V.

VI. BETWEEN the same two Points can be drawn but one right 15
Line.

VII. Two right Lines having two Points common to each other, are 16
only one and the same right Line.

BC and AD have two Points common, as A and B, between
which it is impossible to conceive any more than one right Line;
therefore AB and BA make but one Line.

BA being prolonged cannot extend itself C — A — B — D
elsewhere than towards C, nor AB but towards D, wherefore AD
together with BD makes only one right Line; for there can be but
one right Line between the two Points C and D.

VIII. THEREFORE the Position of a right Line depends only upon two 17
Points.

FOR if a Line be drawn through the two given Points, it will be
the Line required, as appears evident, by the preceeding Proposition.

IX. Two

- 18 IX. Two right Lines which cross, or intersect each other, can meet only at the Point where they intersect.
- 19 X. THE Part of a right Line, is a right Line.
WHEN the Part of a right Line is mentioned, it implies somewhat more than a Point.

S E C T. III.

OF the Circular Line.

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THERE being innumerable Sorts of Curvè Lines, I shall here consider only the Circular Curvè, which next to the right Line, is the simplest and easiest to be understood.

DEFINITIONS.

- 20 I. A Circle is a Line having neither Beginning nor End, and which is in all its Parts equally distant from a Point within the same, which Point is called the Center thereof.
- 21 II. THE Circle considered as a Surface, is all that Space which is contained within the Line above described.
- 22 III. THE Line which bounds the Surface of the Circle, or the Circle considered as a Line, is called the Circumference.
- 23 IV. ALL Lines which cross the Surface and pass through the Center, are called Diameters.
- 24 V. EVERY Line which issues from the Center and terminates at the Circumference, is called a Radius, or Semi-diameter.
- 25 VI. LINES that terminate at the Circumference and which do not pass through the Center, are called Chords.
- 26 VII. THAT Part of the Circumference which is cut off by a Chord Line, is called an Arc.

WHEN a Chord Line passes through a Circle, it divides it into two unequal Portions, which are both bounded by the Extremities of that Chord Line. When the Chord of an Arc is mentioned, it signifies the lesser Arc, unless the other be particularly expressed.

A the Center. B the Circumference. DAC a Diameter.

AC a Radius, or Semi-diameter. EF a Chord Line. Fig. 2.

- 27 VIII. THE Circumference of every Circle is supposed to be divided into three hundred and sixty equal Parts called Degrees, each Degree is divided into sixty lesser Parts called Primes, or Minutes, each Prime, or Minute, into sixty Seconds, and each Second, into sixty Thirds, and so on, ad inf.
- 28 IX. CIRCLES which are described from the same Center, are concentric to each other, and those which have not the same Center, are called Excentrics. Fig. 3.

Z and X

Z and X being described from the same Center A, are Concentrics, but E and F, whose Centers are two different Points B and D, are Excentrics.

EVIDENT Propositions concerning the Circular Line.

I. THE Circumference of a Circle may be described with any given ²⁹
Interval, or Distance. Eucl. 1. Prop. 2 and 3. and Book 4. Prop. 1.

THE Instrument commonly used for describing a Circle, is a Pair of Compasses; with which, it is evident, a Line may be taken equal to a Line given; and of two unequal Lines from the greater may be taken a Line equal to the lesser, which make the three Propositions in Euclid, as quoted above.

II. IN the same or equal Circles, the Chords of equal Arcs are equal, ³⁰
and equal Chords correspond to equal Arcs. Eucl. 3. Prop. 24.

THIS must needs be evidently so, for all the Parts of a Circle being composed in the same regular and uniform Manner, there cannot be any Difference conceived between them.

III. IN the same, or equal Circles, the longest Chords subtend the ³¹
greatest Arcs, and the greatest Arcs are subtended by the longest Chords. Eucl. 3. Prop. 28 and 29.

IV. ARCS containing an equal Number of Degrees are greater or ³²
lesser, according as their corresponding Circles are greater or lesser.

THIS is evident, for the greater the Whole is, the greater ought its Part to be: the third Part of a Yard is greater than the third Part of a Foot.

V. THE Chords of Arcs containing equal Degrees are proportionally ³³
longer or shorter, according as their corresponding Circles are greater or lesser.

VI. THE Diameter of a Circle divides it into two equal Parts, each ³⁴
of which is called its Semi-circumference.

VII. ALL the Radii, Semi-diameters, or Lines drawn from the ³⁵
Center of a Circle to its Circumference being equal, it from hence follows, that all Lines drawn from the Center which are shorter than the Radius, will have their Extremities within the Circle, and such as are longer than the Radius will proceed beyond the Circumference, but such as are of the same Length, will just reach thereto.

VIII. EQUAL Circles have equal Radii. ³⁶

FOR Circles are greater or lesser according to the Length of their Radii.

IX. TWO Circles described with the same Radius and from the ³⁷
same Center, are the same, and no Ways distinct one from the other.

THIS is the same here as of two right Lines, which make only one when drawn between the same two Points.

S E C T. IV.

Of the different Position of two right Lines with respect one to another.

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TH E R E are but three different Positions of two right Lines.
1. Either they may meet together, 2. Or not meet, 3. Or they may cross each other, when they meet together, they may either incline, or not incline towards each other. All which Positions are here considered.

OF PERPENDICULAR LINES.

38 DEFINITION. A Line which stands upon or cuts another Line, so that it inclines not more to one Side thereof than to the other, is called a Perpendicular.

EVIDENT Propositions concerning Perpendicular Lines.

39 PROP. I. If the Top of a Line which stands upon the Middle of another Line, be at an equal Distance from the two Ends thereof, then it does not incline to either Side, but is Perpendicular thereto. Fig. 4.

If A, the Top of the Line AE, which stands upon E, the Middle of BD, is equally distant from B and D, the Extremities of the Line BD, it is then Perpendicular thereupon: this follows from the Idea of a Perpendicular Line, as given by the preceeding Definition.

40 II. If there are two Points in a Line which are equally distant from the Extremities of the Line which it stands upon, every Point in that first Line, will be equally distant from the same Extremities of the second Line.

If the two Points A and E of the Line AE, are equally distant from B and D, all the other Points of AE are equally distant from B and D, for there is no Point in the Line AE, which can be conceived nearer B than to D, without imagining AE to be curved in that Point, and therefore not a right Line, which it was supposed to be.

41 III. If one of the Points of a Perpendicular Line, be equally distant from two other Points of the Line which it stands upon, all its other Points are equally distant from those same two Points.

If AE is Perpendicular upon BD, and one of its Points A or E be equally distant from B and D, the other will be also equally distant from the same Points B and D, for if A be equally distant from B and D, and E is not, then AE will incline more to one Side than to the other; and therefore is not Perpendicular, whereas the Supposition required it to be so.

42 IV. To demonstrate that one Line is Perpendicular to another, it is sufficient to show, that two Points of the one, are each at an equal Distance from two Points of the other.

To demonstrate that AE is Perpendicular upon BD, it is sufficient to prove, that its Points A and E are each at an equal Distance from B and D.

V. IF a Line which is Perpendicular upon another Line be prolonged, 43
its produced Part will also be Perpendicular upon the said Line.

AE is Perpendicular upon BD; its produced Part EC is also Perpendicular upon BD, for it is only one and the same right Line; it cannot possibly be conceived to be otherwise, by reason AC does not bend either towards B or D.

VI. IF one Line is Perpendicular upon another, then are those Lines 44
reciprocally Perpendicular to each other.

IF AC is Perpendicular upon BD, the Line BD is Perpendicular upon AC, for B cannot be conceived to incline more towards A than to C, without imagining D at the same Time inclined most to C; and were it so, than A would incline more towards B than to D, therefore AE will not be Perpendicular upon BD, which is contrary to the Supposition, therefore BD is Perpendicular upon AC, as is also AC upon BD.

PROBLEM. I.

To draw a Line Perpendicular to a given Line, from a given Point 45
without the same. Eucl. 1. Prop. 12.

LET the given Point be K, from which let there be drawn a Line Perpendicular upon Z. 1. From the given Point K as a Center, draw the Arc BC crossing the given Line in the Points B and C, which being both in the Line Z, and in the Circumference of a Circle whose Center is K, they are therefore equally distant from K. 2. From C as a Center, with the Distance CK describe an Arch, and from B with the same Extent, cross that same Arch whose Points of Intersection are K and D, which are equally distant from B and C. 3. Draw a right Line from K to D, then two of the Points of that Line K and D, being by Construction, equally distant from B and C, it must be, as has been proved, * that the Line KD is Perpendicular upon Z; which was to be done.

PROBLEM. II.

To erect, or raise a Perpendicular from a Point given in a Line. 46
Eucl. 1. Prop. 11.

FROM the Point K in the Line Z, let there be raised a Perpendicular. 1. From K as a Center, draw a Circle so as to cross the Line Z in two Points, as at A and B. 2. From these two Points A and B, describe two other Circles with any equal Extent taken at Pleasure, so that the said Circles may intersect each other, as here at D. 3. Draw a Line from the Point D to K, which is the Perpendicular required: For by Construction, D is equally distant from A and B, whose Point K is also by Construction, equally distant therefrom, therefore the Line DK, having two of its Points equally distant from A and B, it is Perpendicular thereupon.†

* Sup. n. 43.

† Sup. n. 43.

When the given Point is at or near the extremity of the given Line, as *A*, take any Point at pleasure as *C*, and open the Compasses to the distance of *AC*, and from *C* as a Center describe a Circle, mark the Point where that Circle crosses the given Line, as here at *B*, and draw through *B* and *C* the Diameter *BD*, then will a Line drawn from *D* to *A* be the Perpendicular required.

COROLLARY.

- 47 HENCE follows the method of dividing a right Line into two equal Parts: Eucl. 1. Prop. 10.

SUPPOSE the given Line be *AB*. from *A* and *B* as Centers, and with the same Extent taken at pleasure describe two Circles which intersect here at *C* and *D*, then by reason that *C* and *D* are equally distant from *A* and *B*, a Line drawn from *C* to *D* will be Perpendicular upon *AB*. * Now the Point *E* common to the two Lines *DC* and *AB*, is equally distant from *A* and *B*, † therefore *AE* is equal to *EB*; consequently the Line *AB* is divided into two equal parts.

- 48 THEOREM. 1. THERE cannot be more than one Perpendicular raised upon a Line from the same Point.

AD is Perpendicular upon the Line *BC*. It is to be demonstrated, that there can be no other Perpendicular raised from the Point *A*, as for Example, *AE* or any other whatsoever,

From *A* as a Center, describe the Circle *BFDEGC*. The Points *B* and *C* are equally distant from *A*, and the Point *D*, where the Circle cuts the Perpendicular, is equally distant from *B* and *C* ‡. Therefore *DB = DC*, the two Arcs *bfd* and *cgd* are also equal, || Consequently *D* is in the middle of the Arc *BFDEGC*. If *EA* is Perpendicular to *BC*, then by the same Reason, *BE = CE*, and *BFDE = CGE*; consequently *E* will be also in the middle of *BFDEGC*, therefore *bfd = bfe*, which is absurd.

- 49 THEOREM 2. A Line let fall Perpendicularly upon the Middle of another Line; passeth thro' all the Points which are equally distant from the Extremities of that other Line.

THE Line *AD* is let fall Perpendicularly upon *A* the middle of *BC*. It must be proved, that it passeth thro' all the Points which are equally distant from *B* and *C*, the extremities of *BC*. If you deny it, and would assert that the Point *E*, which *AD* doth not pass thro', is equally distant from *B* and *C*, then let there be drawn a right Line from that Point *E* to *A*, which will be Perpendicular upon *BC*, seeing that two of its Points *A* and *E* are equally distant from *B* and *C*. † Now it was demonstrated by the preceeding Theorem, that there cannot be drawn two Perpendiculars from the Point *A*, therefore it is not true, that the Point *E* is equally distant from *B*, and *C*.

Theorem.

* Sup. n. 43. † Sup. n. 41. ‡ Sup. n. 42.

|| Sup. n. 31. † Sup. n. 43.

THEOREM. 3. FROM the same Point to the same Line, there can be drawn but one Perpendicular. 50

THE Line AD is Perpendicular upon the middle of BC, I say, that from the Point D, no other Line can be drawn Perpendicular upon BC, for every other Line drawn thereto from D, will fall either on one side or the other of the Point A, suppose at E, then if EA is a Perpendicular, the Point E must be equally distant from B and C, $\parallel$ therefore BE is the half of the Line BC, but AB is also the half, therefore $BA = BE$, which is absurd therefore, &c.

THEOREM 4. Two Lines which are Perpendicular upon a third, can never meet. 51

FOR if they could meet or intersect, there would be at that Point of meeting or intersection, two Perpendiculars upon the same Line, which has been demonstrated to be impossible.

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THE Distance between a Point and a Line, is measured by a Perpendicular, because it is the simplest and most certain Measure, by reason there can be but one Perpendicular drawn from a Point to a Line, and besides, it is the shortest of all lines which can be drawn from the same Point to the same Line, as will be made appear by the following.

THEOREM 5. THE Perpendicular is the shortest of all Lines which can be drawn from the same Point to the same Line. 52

THE Line BA is Perpendicular upon Z, It is to be demonstrated, that it is shorter than any other drawn from the Point B to the Line Z. Prolong BA unto C, making $BA = AC$, the Line DA is Perpendicular upon BC, as is BC likewise upon AD * The Point D is therefore equally distant from B and C, $\dagger$ therefore BD is equal to DC; But the right Line BC is shorter than the Line $BD + DC$ $\ddagger$ consequently AB, the half of BC, is shorter than BD, the half of $BD + DC$, Q. E. D.

OF OBLIQUE LINES.

DEF. 1. A Line which meets another Line, so as to incline more to one side thereof than to the other, is called an Oblique Line.

DEF. 2. BC is an oblique Line upon Z; having drawn from its extremity B, the Perpendicular AB, the Line AC which is between A, the Foot of the Perpendicular, and C that of the Oblique, is the distance between the oblique Line and its Perpendicular, and that distance is the measure of the Obliquity of BC. Therefore a Line is more or less oblique, in proportion to the distance thereof from its Perpendicular.

LEMMA. 1. Two right Lines having upon each of them a Perpendicular, if they be placed one upon the other, so as the foot of one Perpendicular be directly upon that of the other, the two Perpendiculars will coincide together.

AB is

$\parallel$ Sup. n. 42. * Sup. n. 42. $\dagger$ Sup. n. 45. $\ddagger$ Sup. n. 12.

AB is Perpendicular upon Z , and ab upon X , 'tis to be proved. That if X be laid upon Z So that a be put upon A , the two Perpendiculars AB and ab will coincide, for since that after the above mentioned Position, X and Z become only one Line, AB will be Perpendicular upon each of them, so also will ab , therefore if AB and ab did not coincide, there would be two Perpendiculars upon the same Line at the same Point, which is impossible.*

LEMMA. 2. IF from the Ends of a Line be drawn four Lines, two of which unite in a Point nearer the given Line than the other two, I say, that the two last taken together will be longer than the two first.

LET BC be the Line given, and from the Points B and C , draw the two Lines BD and CD , and the two others BE , CE , I say, that $BE + CE$ is longer than $BD + CD$. A right Line being the shortest between two points. $+ CE + EG$ is longer than $CD + DG$, and by the same Reason $BG + GD$ is longer than BD , therefore $CE + EG + GD + BG$ is longer than $CD + DG + BD$, taking DG away from each, the remainder $CE + EG + BG$ or $CE + EB$, will, according to the seventh Axiom, be longer than $BD + DC$, which was to be proved.

THEOREM I. WHEN the Perpendiculars and their Obliquities are equal, the Oblique Lines are equal.

AB is Perpendicular upon AC , and ab upon ac , the two perpendiculars are equal, also AC the distance of the Perpendicular AB , is equal to ac , the distance of the Perpendicular ab . it is to be proved, that $BC = bc$. Having according to Lemma 1. Placed the two equal Lines ac , and AC upon each other, the Perpendicular ab will coincide with the Perpendicular AB , and c with C , so likewise bc with BC , Q. E. D.

THEOREM. 2. OBLIQUE Lines drawn from the same Point to the same Line, are in length Proportional to their distance from the Perpendicular.

It is to be proved, That BE is longer than BD , to do which, let BA be prolonged even to C so that $AB = AC$, then $+ BD = DC$, and $BE = EC$, now $BE + EC$ is longer than $BD + DC$ by Lemma 2; therefore BE the half of $BE + EC$, is longer than BD the half of $BD + DC$, according to Axiom 9.

THEOREM. 3. AN Oblique Line is longer, according to the Length of its Perpendicular and its Distance therefrom; and if its Distance from the Perpendicular be the same, the longer the Oblique Line is, the longer is its Perpendicular.

1. LET BC and DE be two Oblique Lines, the Perpendicular AB is longer than AD , and AC its Distance from the Perpendicular BA , is longer than AE that of DA , I say, that the Oblique Line BC is longer than the Oblique DE , for (by the preceding Theorem.) BE is longer than BA , and since that AC is Perpendicular upon AB , by the same Theorem, BE is longer than DE , consequently, BC being longer than BE , is also longer than DE . 2. AE is the same Distance, I say, that if BE is longer than DE , the Perpendicular AB is longer.

* Sup. n. 49.

† Sup. n. 12.

‡ Sup. n. 38.

is longer than the Perpendicular AD, for if AD were equal or longer than AB, then; either by the first or precedent Theorem, BE would be shorter than DE, or else equal thereto, contrary to the Supposition which required BE to be longest.

THEOREM 4. *If the Perpendiculars and Oblique Lines are equal, their Distances from the Perpendiculars are equal.*

LET $AB = ab$, and $BC = bc$, It must be proved, that $AC = ac$, Having placed ab upon AB , the two equal Lines will entirely coincide, and by Lemma 1, the Perpendicular ac will by this Means partly coincide with AC . If AC is shorter than ac , and that c does therefore not coincide with C , but with D , then bc will coincide with BD , but $BD > BC$, * which is contrary to the Supposition $BC = bc$, 'twould have been an equal Absurdity to have supposed AC longer than ac , therefore it must be, that $AC = ac$, Q. E. D.

THEOREM 5. *If the Oblique Lines and their Distances from the Perpendiculars are equal, the Perpendiculars are equal.*

LET $AC = ac$, and $BC = bc$, It is to be proved, that $AB = ab$, Let AC be placed upon ac , the two equal Lines will coincide, and the Perpendicular ab with the Perpendicular AB in part at least, by Lemma the first, † If it is said that $AB < ab$, and that therefore b coincides with D , then bc will coincide with DC and is therefore equal thereto, and therefore DC † longer than BC , which was supposed equal, the Conclusion would have been equally absurd, if AB had been supposed longer than ab , therefore ab must entirely coincide with AB , and therefore $AB = ab$, which was to be proved.

OF PARALLEL LINES.

DEFINITION. *Two right Lines which are equally distant in all their Parts one from the other, are called Parallels.*

EVIDENT Propositions concerning Parallel Lines.

PROP. I. *A right Line which hath two of its Points equally distant from another right Line, is parallel thereto.*

FOR the Position of a right Line depends only on two of its Points, therefore the two Lines being equally distant from each other, they are parallel, according to the Definition.

PROP. II. *PERPENDICULARS which are between two Parallels are equal.*

THE Perpendiculars measure the Distance between the two Parallels, which being every where the same, they are equal.

PROP. III. *Two parallel Lines will not meet, supposing them prolonged, ad inf.*

FOR they cannot meet unless they approach at one end, they are therefore then no longer at the same Distance, consequently cease to be (as supposed) parallel.

PROP. IV. *Two right Lines which are not parallel, but nearer at one end than the other, if sufficiently prolonged will meet.*

ADVER.

* Sup. n. 59.

† Sup. n. 56.

‡ Sup. n. 59.

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THIS is to be understood only of right Lines, for there are Curved Lines of such a Nature, that they continually approach toward right Lines without ever meeting them, (as is demonstrable) and these right Lines are called the Asymptotes of the Curves.

68 PROP. V. TWO Lines which are Perpendicular upon one and the same Line, are parallel one to another.

FOR two Perpendiculars can never meet.* Now if they were not parallel, they would meet according to the preceding Proposition. therefore they must be parallel.

69 LEMMA 1. LINES which are between two parallel Lines, if they are Perpendicular to one parallel, they are so likewise to the other.

IF AB Perpendicular upon X is not so upon Z, then Z is not so upon AB; * therefore being inclined to the Line AB, it will approach either towards one end or the other of the Line X, and meet it, † consequently it is not parallel thereto, contrary to the Supposition which required it to be so.

70 LEMMA 2. THE Line AB cannot be Perpendicular upon both Z and X, unless the two Lines are parallel. §

FOR the two Lines (same Figure) are reciprocally Perpendicular upon AB; † and therefore they are parallel.

71 LEMMA 3. IF between two right Lines there are two other equal right Lines, of which one is Perpendicular upon the first, and the other upon the second, I say, that the two first Lines are parallel.

BETWEEN Z and X, are two equal Lines AB and CD, one of which, namely AB is Perpendicular upon X, and CD upon D, I say, that Z and X are parallel. For if Z is not parallel to X, it either declines from it, or inclines to it, both which shall be shewn to be absurd, therefore it must be that they are parallel. 1. If it be supposed to decline therefrom, From the Point A having drawn AE Perpendicular upon Z, * then if this Perpendicular coincide with AB, Z and X are Perpendicular upon AB by Construction, and therefore parallel to each other † contrary to the Supposition, That if AE fall on one side of AB, the Perpendicular AE will be shorter than the Oblique Line AB, ‡ consequently shorter also than CD = AB, § therefore the Line Z inclines to the Line X, contrary to the Supposition. 2. If it is supposed that the Line Z inclines to the Line X, 'twill be a like Absurdity; for having raised upon the Line Z the Perpendicular BE at the Point B, || as AB is Perpendicular upon X, if it coincides with AB, then according to what has been said, Z and X are parallel, but if BE falls on one side, as AB was proposed Perpendicular upon X, BE will be Oblique, and consequently longer than BA, ¶ or than its equal CD, therefore Z and

* Sup. n. 52.

• Sup. n. 45.

• Sup. n. 46.

† Sup. n. 67.

† Sup. n. 68.

|| Sup. n. 47.

‡ Sup. n. 45.

‡ Sup. n. 53.

¶ Sup. n. 51.

§ Sup. n. 68.

§ Sup. n.

Axiom 3.

and X decline from each other, contrary to the *Supposition* which made them incline, Q. E. D.

PROBLEM. To draw a Line parallel to a Line given through a given Point. Eucl. 1. Prop. 31.

LET AD be the Line to which a Parallel must be drawn through the Point C, From the Point C let fall a Perpendicular upon the given Line AD, upon which having raised a Perpendicular such as AB equal to CD; it is evident, that the Line which passeth through the two Points B and C will be parallel to AD, according to the Definition.*

OR from any Point whatsoever as A, let such an Arc be described as will pass through the given Point C, from which with the same extent AC, having made the Arc AB equal to DC, the Line drawn through the Points B and C will be the parallel required.

THEOREM. Two Lines parallel to a third, are parallel to each other. Eucl. 1. Prop. 30.

X and Y are parallel to Z. From X draw AB Perpendicular upon Z, which being prolonged to C, seeing that it is Perpendicular upon Z it will be so upon Y, (a parallel to Z; †) and since Z is a parallel to X, the Line which is Perpendicular upon Z will be so likewise upon X, (a parallel to Z. †) therefore since X and Y are Perpendicular upon AC, they are parallel to each other. § Q. E. D.

COROLLARY. Two different Lines which are parallel to a third, cannot be drawn through the same Point.

FOR by the Theorem they must be parallel to each other, which since they would have one Point common is absurd, and it is the Property of Parallels never to meet.

SECT. V.

OF the different Position of two Circles with respect to each other.

ADVERTISEMENT.

ONE Circle may be so placed with respect to another Circle, 1. THAT they may neither intersect nor touch. 2. That they may intersect or not intersect. 3. That they may touch either on the inside or outside.

EVIDENT Propositions concerning the Position of Circles.

PROP. 1. Concentric-Circles do neither intersect nor touch.

THE Concentric-Circles X and Z whose Center is A, may be conceived as described by the Points D and E of the Line AE, therefore the Circle which D describes, is always within the Circle

D

* Sup. n. 63. † Sup. n. 69. ‡ Sup. n. 69. § Sup. n. 68.

cle Z, which is described by E, the two Circles therefore cannot meet.

- 76 PROP. 2. Two Concentric-Circles are always equal distant.
FOR between X and Z there is always the same distance DE.
(same fig.)

- 77 THEOREM 1. Two Circles which intersect are not Concentric.
Eucl. 3. Prop. 5.

LET there be two Circles which intersect at the Points B and E, I say, that they have not the same Point for their Center. For if A be the Center of the two Circles which intersect at the Point B, the Lines AB and AC (Radii of the same Circle) are equal, * $AC = AB$. and by the same reason $AB = AD$. therefore $AC = AB = AD$; then $AC = AD$, † that is, the Part equal to the Whole, which cannot be. †

- 78 THEOREM 2. Two Circles which touch are not Concentric, or have not the same Center. Eucl. 3. Prop. 6.

LET there be two Circles which touch at the Point B, I say, they have different Centers. For if A was the Center of both the Circles, then $AD = AB$, and $AB = AC$, § consequently $AD = AB = AC$, therefore $AD = AC$, || that is, the Part AC is equal to its Whole AD, which is absurd.

- 79 THEOREM 3. IF two or more Circles have their Center in the same right Line, each crossing it at the same Point, they can touch each other only at that Point.

THE Circles X, Z, Y, have their Centers in the same Line which they cross at the Point D. it is to be proved, that they can touch only at the Point D. If we should say that they may touch elsewhere, or meet at the Point E, then $AE = AD$ seeing they are the Radii of the same Circle, by the same reason $CD = CE$, therefore $AE + CE = AD + CD$, which is absurd.* In like manner it is demonstrated that Z and X do not meet in E, for if they so met, $BD = BE$, and $AE = AD$, but $AD = AB + BD$, or $AB + BE$, then $AE = AB + BE$, which is absurd.†

- 80 COROLLARY. Two Circles cannot touch either within side, or without, but only in one Point. Eucl. 3. Prop. 13.

X and Z touch within side, (same Figure) if it be at the Point D, they cannot touch elsewhere, for Example at the Point E, if they touch at E they can't touch at D, for $AB + BE$ will be equal to AE , which is absurd.† X and Y touch on the out-side, if it be at the Point D, they cannot touch in any other, for Example in E, for then $AC = AE + EC$, which is absurd.||

- 81 THEO. 4. IF two Circles touch on the inside, the right Line which being produced will join their Center, will pass through their Point of Contact. Eucl. 3. Prop. 11.

1. By Theorem 2. These two Circles have not the same Center. 2. Let F be the Center of BAB, if G which is not in the Line FA, which passeth through the Point of Contact, was the Center

* Sup. n. 20. † Sup. Ax. 3. ‡ Sup. Ax. 1. § Sup. n. 20.
|| Sup. Ax. 3.

* Sup. n. 1. † Sup. n. 12. ‡ Sup. n. 12. || Sup. n. 12.

Center of DAD; then $GA = GD$, * therefore $FG + GA = FG + GD$, †
 Now $FG + GA > FA$ † and consequently longer than FB , for
 $FA = FB$, ‖ therefore $FG + GD > FB$ or $FG + GB$, then take from
 each the common Part FG , there will remain $GD >$ than the
 Whole GB , § *which is impossible.*

THEOREM 5. IF two Circles touch on the out-side, a right Line ⁸²
 drawn through their Centers, will pass through their Point of Contact.
 Eucl. 3. Prop. 12.

LET DAD and EAE be the two Circles which touch at A,
 say, that the Line which will join their Centers will pass thro'
 the Point A. If we deny this, we must unavoidably fall into an
 Absurdity; for let B be the Center of DAD, and C that of EAE,
 and that therefore the Line BC will not pass through A where
 the two Circles touch: $BA = BD$ and $CA = CE$, * then $BA +$
 $AC = BD + CE$, since that BC does not pass through the Point
 of Contact of the two Circles which is common, D and E are
 not the same Point, there being between them the Distance DE,
 which add to $BD + CE$, then $BD + DE + CE$ is longer than
 $AB + AC$, *which is absurd.* †

S E C T. VI.

OF the Position of a right Line with respect to a Circle.

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A RIGHT Line may be wholly within, or without a Circle:
 If it is without, it may be in such Position, that when pro-
 longed it may either cross it, or only touch without entering
 into it.

THEOREM 1. IF any two Points be taken in the Circumference of ⁸³
 a Circle, the right Line drawn from one of those Points to the other,
 will fall wholly within the Circle. Eucl. 3. Prop. 2.

B and C are two Points in the Circumference of the Circle X,
 it must be proved, that the Line BC drawn between those Points,
 is entirely within the Circle. From A the Center of X, draw a
 right Line to D the Middle of BC; since AB and AC, the Ra-
 dii of X, are equal, and since D is the Middle of BC, the Line
 AD is Perpendicular, † and therefore shorter than either AB or
 AC, ‖ therefore D is within the Circle. § Now every other obli-
 que Line drawn from A to BC, will also be shorter than either
 AB or AC, * all the Line BC is therefore within the Circle.

D 2

THEOREM 2.

- | | | | |
|---------------|---------------|---------------|---------------|
| * Sup. n. 20. | † Sup. Ax. 4. | ‡ Sup. n. 12. | ‖ Sup. n. 20. |
| § Sup. Ax. 7. | | | |
| * Sup. n. 20. | † Sup. n. 12. | ‡ Sup. n. 36. | ‖ Sup. n. 40. |
| § Sup. n. 53. | | | |
| * Sup. n. 59. | | | |

- 84 THEOREM 2. If a Chord is cut into two equal Parts by a Perpendicular Line, I say, that Line cuts the Arc of the Circle into two equal Parts, and passes through the Center.

LET the Chord CD be cut into two equal Parts by the Perpendicular Line BF, at the Point E, I say, that this Line cuts the Arc CD equally in two at B, and passes through the Circle's Center. 1. Seeing that the Point E is equally distant from the Points C and D, and that BF is Perpendicular upon CD, the Point B which is in this Perpendicular, is also equally distant from the Points C and D, † and therefore the Chord BC is equal to the Chord BD, and consequently the Arc BC is equal to the Arc BD, ‡ therefore the Arc CBD is cut equally in two, it is the same with the Arc CFD. 2. The Perpendicular BF passeth thro' all the Points which are equally distant from C and D, || now the Center of the Circle is equally distant from these two Points C and D, therefore BC passeth through the Center.

- 85 COROLLARY 1. IT is evident, that in order to divide an Arc into two equal Parts, a Perpendicular must be rais'd upon the Middle of its Chord Line. Eucl. 3. Prop. 30.

- 86 COROLLARY 2. THE two Arcs contained between two parallel Lines are equal.

THE two Arcs MC and ND, contain'd between the two parallel Lines or Chords MN and CD are equal; for the Diameter BF being drawn Perpendicular upon CD, * it will also be so upon MN, † therefore BM=BN, and BC=BD, ‡ the Arcs of equal Chords are equal, || therefore the Arc BC—BM=BD—BN; but the Arc BC less the Arc BM is equal to the Arc CM, and likewise BD—BN=ND; therefore CM=ND, Q. E. D. but if instead of the Chord MN, we had supposed at B, a Tangent parallel to CD, the Arc BC, might more readily have been demonstrated equal to the Arc BD.

- 87 PROBLEM 1. THREE Points being given, to find the Center of the Circle which passes through those Points. Eucl. 3. Prop. 1.

LET the three given Points be A, B, C, join them by two right Lines AB, BC, which we will consider as Chords of the Circle sought, and on the Middle of each of those two Lines draw two Perpendiculars, I say, that the Point K, where those two Lines intersect, is the Center of the Circle required, which is evident by the foregoing Theorem.

IF the three given Points had been in a right Line, the Problem would have been impossible, as is evident, for then the two Perpendiculars being parallel, could not intersect.

- 88 COROLLARY 1. IF two Circles have three Points common, as A, B, C, then the rest are all likewise common.

THESE two Circles having the same Center (viz.) K, and being described with the same Extent, they cannot be different from each other.*

COROLLARY 2.

† Sup. n. 42.

‡ Sup. n. 31.

|| Sup. n. 42.

* Sup. n. 46.

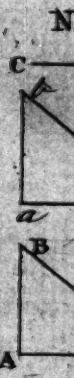
† Sup. n. 69.

‡ Sup. n. 42.

|| Sup. n. 31.

* Sup. n. 38.

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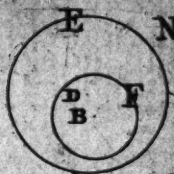


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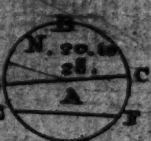
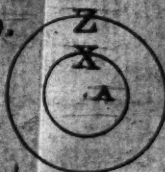


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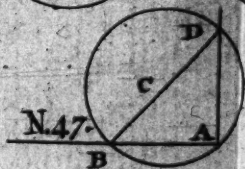
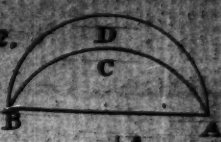




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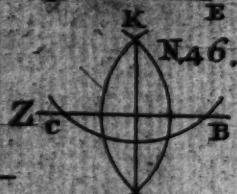
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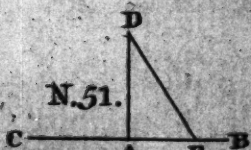


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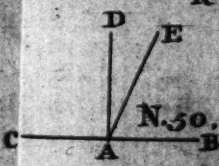


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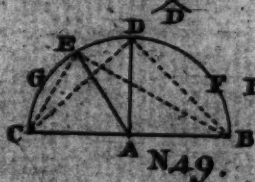
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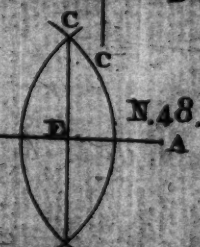
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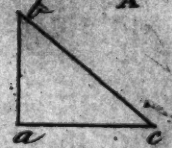
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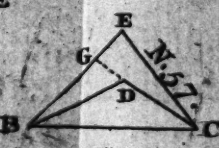
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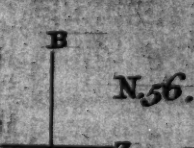
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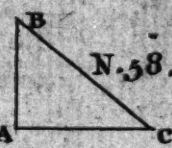
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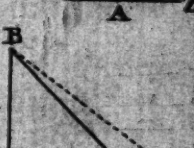
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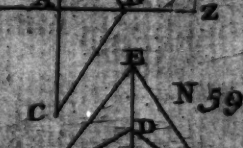
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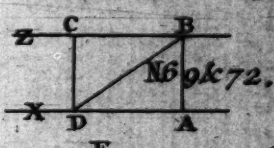
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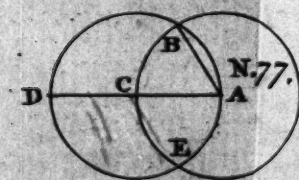
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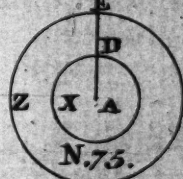
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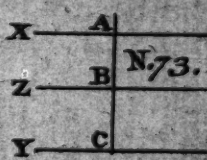
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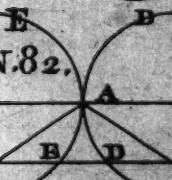
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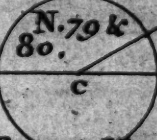
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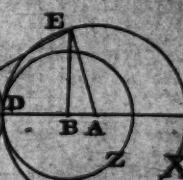
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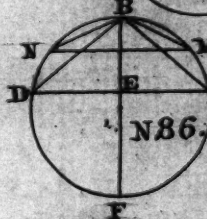
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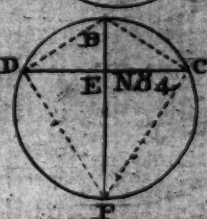
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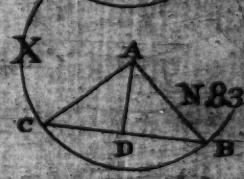
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N.83.

COROLLARY. 2. Two Circles can intersect only in two Points, 89
Eucl. 3. Prop. 10.

For if they could intersect in three, they would have three Points common, therefore, by the preceding Corollary, they would not be two different Circles.

COROLLARY. 3. PART of a Circle being given, the whole Circle may be completed. Eucl. 3. Prop. 25. 90

MARK three Points in the Part given, then may the Center of the Circle whose part it is, be found by the preceding Problem.

THEOREM. 3. IF a Line cuts the Chord or Arc of a Circle into two equal Parts, and passeth thro' the Center, I say, that it cuts it Perpendicularly. Eucl. 3. Prop. 3. 91

LET BK be the Line which cuts the Arc or Chord of a Circle into two equal Parts, and passes thro' the Center K. I say, that it cuts the Chord Perpendicularly, for in the Line BK there are two Points (*viz.*) A or B and K equally distant from C and D, since AD is half the Line BD, and BC half the Arc CBD, and that K is the Center, therefore BK is Perpendicular. *

THEOREM. 4. IF a Line cuts the Chord of a Circle Perpendicularly, and passes thro' the Center, I say, that it divides it into two equal Parts. Eucl. 3. Prop. 3. 92

THE Line BK passes thro' the Center K, and is Perpendicular upon CD, I say, that it cuts CD equally in two.

THE Center is equally distant from C and D, and BK being Perpendicular, the Point A, and all the other Points of BK, ought to be equally distant from C and D. § Then $AC=AD$, therefore CD is divided into two equal parts.

THEOREM. 5. TWO Chords which don't pass thro' the Center, cannot intersect so as to divide each other into two equal parts. Eucl. 3. Prop. 4. 93

LET DE and BC be two Chords which intersect out of the Circle's Center at the Point A, I say, that they are unequally divided. If the two Chords intersect at A, which is not the Center, and that this Point be the Middle of the two Lines, then having drawn from A, a Line to K the Center, that Line KA, will be Perpendicular upon BC and upon DE. † Then BC and DE are Perpendicular upon KA, ‡ therefore, there is two Perpendiculars upon the same Point A, which is impossible. || 94

THEOREM. 6. CHORDS which are equally distant from the Center are equal, and if they are equal, they are equally distant from the Center. Eucl. 3. Prop. 14.

LET BC and GH be Chords equally distant from the Center K. I say, that they are equal, and if they are equal, their distances KE and KF from the Center are equal. 1. Upon these Chords I draw the Perpendiculars DK and KL, which intersect at the Middle. † by the Hypothesis, $KE=KF$, and since that $BK=KH$, and $KC=KG$, then the Oblique Line BK being equal to

* Sup. n. 40. § Sup. n. 42. † Sup. n. 91.
‡ Sup. n. 45. || Sup. n. 49. ‡ Sup. n. 92.

to the Oblique KH, and the Perpendiculars KE and KF belonging to those Obliques being equal, BE and HF their distances from the Perpendiculars are also equal. * the same Way it is proved, that $EC=FG$, and that therefore $BC=HG$, which was to be proved. 2. It has been proved, that when Oblique Lines as KB and KH are equal, and that BE and HF their distances from the Perpendicular are also equal, that then the Perpendiculars KE and KF are equal. ¶

95 THEOREM. 7. THE Diameter, or Line which passeth thro' the Center, is the longest of all Lines which are in the Circle. Eucl. 3. Prop. 15.

LET the Line AB be the Diameter. It must be proved, that it is longer than CD, or any other Line whatsoever which doth not pass thro' the Center, which is plain. For $KC=KB$, and $KD=KA$, therefore $BA=KC+KD$, now $KC+KD$ is longer than CD, † therefore BA is longer than CD.

96 THEOREM. 8. CHORDS nearest to the Center of the Circle are longest, and the longest Chords are nearest the Circle's Center. Eucl. 3. Prop. 15.

LET A be the Center, and the proposed Lines be BC and DE, it must be proved, that DE which is nearest the Center A, is longer than BC which is farther off. AM is the Distance of BC, and AN that of DE, the latter of which is Shortest, it must be proved, that DE which is nearest the Center, is longer than BC which farther off, which is evident, for the Arc EFD is longer than the Arc CFB. Now the greatest Arcs have the longest Chords, § therefore DE the Chord of EFD, is longer than BC the Chord of CFB, Q. E. D.

97 THEOREM. 9. If a Line is the common Chord of two Arcs of unequal Circles which intersect, I say, that the Arc of the lesser Circle contains more Degrees than the Arc of the greater.

LET X and Z be two unequal Circles which intersect at the Points C and D, their Chord CD is common; I say that the Arc CZD of the lesser Circle, contains more Degrees than the Arc CXD of the Greater, for if the two Arcs of which CD is the Chord were the Same, each for Example of ten Degrees, then what has been before proved ‡ would be false, (Viz.) that the Chords of Arcs containing the same Number of Degrees, are proportionally Longer as their corresponding Circles are greater.

98 COROLLARY. THEREFORE the same Line cannot be the Chord of two Arcs containing the same Number of Degrees, and which are portions of unequal Circles.

IT was observed, Sup. n. 26. that the lesser Arc of the Circle is to be understood, unless the other is directly mentioned.

99 THEOREM 10. IF from a Point without the Circle, be drawn several Lines which cross it, and end at the Circumference, I say, 1. Of all the Lines which fall upon the convex Part, that which being

* Sup. n. 61. ¶ Sup. n. 62. † Sup. n. 12.

§ Sup. n. 32. ‡ Sup. n. 33.

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being produced will pass thro' the Center, is the Shortest, next, those which are nearer that, are shorter than those which are farther off. 2. It is just the Reverse, of the Lines which fall upon the Concave Part. Eucl. 3. Prop. 8.

LET the Point be B, from which having drawn the Lines BH. BG. BF. I say, 1. That BC which passeth thro' the Center, is shorter than any other drawn from the same Point, than BD for Example. For $AC=AD$, now $AC+CB$ is shorter than $AD+DB$, * therefore BC is shorter than BD $AD=AE$. Now $AD+DB$ is shorter than $AE+EB$, † then taking away the equal Quantities AD and AE, the Remainder DB is shorter than the Remainder BE. ||

2. THAT BH which passeth thro' the Center is longest, for $AB+AH=AB+AG$, now $AB+AG$ is longer than BG § now DG is longer than DF. † therefore BH is longer than BG. it is likewise evident, that $BD+DF$ is longer than BF, || therefore $BD+DG$ is longer than BF.

COROLLARY. IF there are two right Lines, equal or unequal, placed one upon the Other and which agree to a Point at one of their Ends, upon which Point if one of them be turned round, the Line which joins their other ends, will as that Line moves, continually increase its length, even till the said two Lines, make only one right Line. 100

LET AB, and AC be the two Lines upon AB one of which as AC, turn round upon the End A, in moving from AB it will pass thro' D, E, F. The Lines which join their ends as BD, BE, BF increase in length even till the said two Lines form only the right Line BAH which is what was to be proved.

THEOREM. 11. IF there be Lines drawn from a Point which is out of the Center of a Circle, to the Circumference, I say, that which passeth thro' the Center is the Longest, and the other Part remaining which will compleat the Diameter, is the shortest Eucl. 3. Prop. 7. 101

LET the Point be B, and let there be drawn thro' this Point the Lines BC, BF, BD, and BE, which passes thro' the Center A, I say, that BE is the longest, and its part BD the shortest of all those which end in that Point. 1. $BA+AE=BA+AC$, seeing that $AE=AC$, now $AB+AC > BC$, * therefore BE equal to $BA+AC$ is longer than BC. 2. $AF=AB+BD$, now $AB+BF$ is longer than AF, ¶ then taking away the common part AB, the Remainder BF is longer than the Remainder BD. §

THEOREM. 12. THERE is no Point in a Circle, except the Center, from which there can be drawn to the Circumference any more than two equal Lines. Eucl. 3. Prop. 9. 102

LET A be the Center of the Circle X, therefore the Point B is not the Center. 1. I say, that BD is longer than BC, † 2. That BE is longer than BD, for AD and AE are each the Radii of X, which in turning or removing from AB, ought to make BE longer

* Sup. n. 12. † Sup. n. 57. || Sup. Ax. 7.

§ Sup. ax. 7. † Sup. n. 96: || f. f. n. 12.

* Sup. n. 12. ¶ Sup. n. 12. § Sup. Ax. 7. † Sup. n. 101.

BE longer than BD,† and in like Manner BF is longer than BE, now BG is longer than BF,* Consequently, all Lines drawn from B to any part of the Semi-circumference of X, as CD, FG, &c. are unequal. Let us take the Arc CH equal to CD, since that AC is Radius, it will cut DH Perpendicularly,‡ and in two equal Parts at the Point I,§ therefore the Perpendicular and its Distance from the Oblique Line being equal, the Oblique Lines BH, BD are equal. || But every other Line except BH, will as has been proved, be either shorter or longer, in the same Manner it may be demonstrated, that there can be drawn but two equal Lines from C.

103 COROLLARY 1. HENCE, it is only from the Center of the Circle, that there can be drawn more than two equal Lines to the Circumference.

104 COROLLARY 2. IN every Circle, that Point from which three equal Lines may be drawn to the Circumference, is the Center of the Circle. Eucl. 3. Prop. 9.

105 THEOREM 13. A Line which crosses two concentric Circles, whether it pass through their Center or not, the Parts of the Line which are between the Circumferences of those two Circles, are equal to each other.

LET BC, BD cross two concentric Circles, I say, that $BM=NC$ and $BF=ED$. 1. For BC which passeth through the Center; this has been proved.* 2. From A draw a Perpendicular upon BD, then $GD=GB$, and $GE=GF$,† then $GD-GE=GB-GF$ ‡ now $GD-GE=DE$, and $GB-GF=FB$, therefore $DE=BF$.

106 THEOREM 14. A Line Perpendicular at the End of Radius touches the Circle, and that in only one Point. Eucl. 3. Prop. 16 and 18.

BD is Perpendicular upon BK, it must be proved that this Line touches the Circle X only at the Point B. If you say that it touches it in a second Point, as in C; draw a Line from K to C, which is not Perpendicular upon BD; since by reason there can be drawn but one Perpendicular from K upon BD; § it is therefore longer than the Radius BK, which is Perpendicular upon BD,|| wherefore the Point C is out of the Circle X, therefore BD does not touch in the two Points B and C, but in one only, as B.

107 COROLLARY. A Circle cannot be touched by any more than one right Line, at the same Point.

Two different Lines cannot touch the Circle X at the same Point B; for if so, they by the Theorem, they would both be Perpendicular upon BK, which is impossible.*

108 THEOREM 15. IF a Line be drawn within a Circle perpendicularly upon the Point of Contact of the Tangent or touching Line, it passeth through the Center of the Circle. Eucl. 3. Prop. 19.

CD is

† Sup. n. 100. * Sup. n. 101. ‡ Sup. n. 91. § Sup. n. 42.

|| Sup. n. 58.

* Sup. n. 76. † Sup. n. 92. ‡ Sup. Ax. 5. § Sup. n. 51.

|| Sup. n. 53.

* Sup. n. 49.

CD is a Tangent or touching Line, from C the Point of Contact, draw a Perpendicular within the Circle, I say, that it will pass through the Center K. If you think it passes through B which is not the Center, I prove it false, having drawn from C the Radius KC, and raised at C a Perpendicular which will be a Tangent,† but it will be the same as CD, since there can be only one Tangent at the Point C, ‡ therefore at the same Point upon the Line CD, there would be two Perpendiculars CK and CB, which is impossible. †

THEO. 16. *There cannot be drawn, a right Line between a Tangent and the Circumference of a Circle, but there may be drawn an infinite Number of circular Lines.* Eucl. 3. Prop. 16. 109

IF between the Tangent BD and the Circle X, there could be drawn a right Line, dividing the Space between the Tangent BD and the Circle, as the Line BF, upon that Line from the Point K, draw another Line Perpendicular thereto, to wit KE, which will be shorter than the Radius BK which is not Perpendicular upon that Line,* therefore KE being shorter than the Radius BK, its Extremity E is within the Circle, consequently the Line BF cuts the Circle, and does not therefore divide the Space which is between it, and the Tangent BD.

BUT between the Tangent AB and the Circle Y, may be drawn an infinite Number of Circles, for having produced the Radius AC beyond the Center C, and from E as a Center, with the extent of EA having made the Circle Z, the Line AB will be Tangent thereto;† which Circle Z, being greater than Y, will be out of the Circle Y, in like Manner the Circle X whose Center is F, will be also between AB and Y, &c. *ad inf.* consequently between the Tangent AB and the Circle Y, an infinite Number of circular Lines may be drawn.

COROLLARY. *It is therefore evident, that the Space contained between the Tangent and the Circumference of a Circle, may be divided into an infinite Number of Parts.* 110

THEOREM 17. *FROM a Point which is without the Circle, there can be but one Tangent drawn to the Circle on the same side thereof.* 111

ALL other Lines will either intersect it or not reach thereto. Let A be a Point given out of the Circle, the Line AB touches it at the Point B, I say, that there can be no other Tangent drawn from A towards B, but that all other Lines will either intersect, or else not touch. 1. If a Line be drawn beyond AB, it will not touch the Circle, and 2. If it be drawn through the Point B, it is the same as AB. 3. And if it do not reach so far as the Point B, but goes through D, because CB is longer than CD, then the Point D will be in the Circle, * therefore AD enters within the Circle, and cuts it.

E

PROBLEM

† Sup. n. 106.

* Sup. n. 53.

* Sup. n. 36.

† Sup. n. 107.

† Sup. n. 106.

† Sup. n. 51.

112 PROBLEM 2. To draw a right Line so as to touch a Circle in a given Point.

THE Center is K, the given Point B, draw the Radius KB, and upon the End B,† raise the Perpendicular BD which will be the Tangent required.†

113 PROBLEM 3. To draw a Tangent, from a Point given out of a Circle. Eucl. 3. Prop. 17.

THE Circle is BEB, the given Point is C, from which to the Center A draw a Line, and at the Point B where that Line intersects the Circle, by the preceedent Problem, draw the Tangent GD, thro' C describe a concentric Circle, and from D where that Circle is cut by the Tangent GD, take DF equal to DC, and join C and F by a Line, which will be the Tangent required. By Construction the Chord $GD = CF$, for the Arc GC is equal to the Arc CD,* and the Arc CD to the Arc DF, therefore the Arcs GD and CF being equal, their Chords are equal.† From D to the Center A draw the Line AD, which will be Perpendicular upon CF, seeing two of its Points, (viz.) A and D, are equally distant from its Ends, now since the Chords DG and CF are equal, the Lines AB and AE are equal,‡ then the Point E as well as B, is in the Circumference of the Circle BEB, therefore the Line CF being Perpendicular upon E, the End of the Radius AE, it touches the Circle.§

OR more easily thus, let A be the given Point, from which a Tangent must be drawn to the Circle X, after having drawn the Line AB, from A to B, the Center of the Circle X, the Semi-circle ABC must be described upon that Line, and at C the Point of Section, draw AC which will be the Tangent required, the Demonstration of which is not proper here.

† Sup. n. 47.

* Sup. n. 92.

§ Sup. n. 106.

† Sup. n. 106.

† Sup. n. 31.

‡ Sup. n. 94.



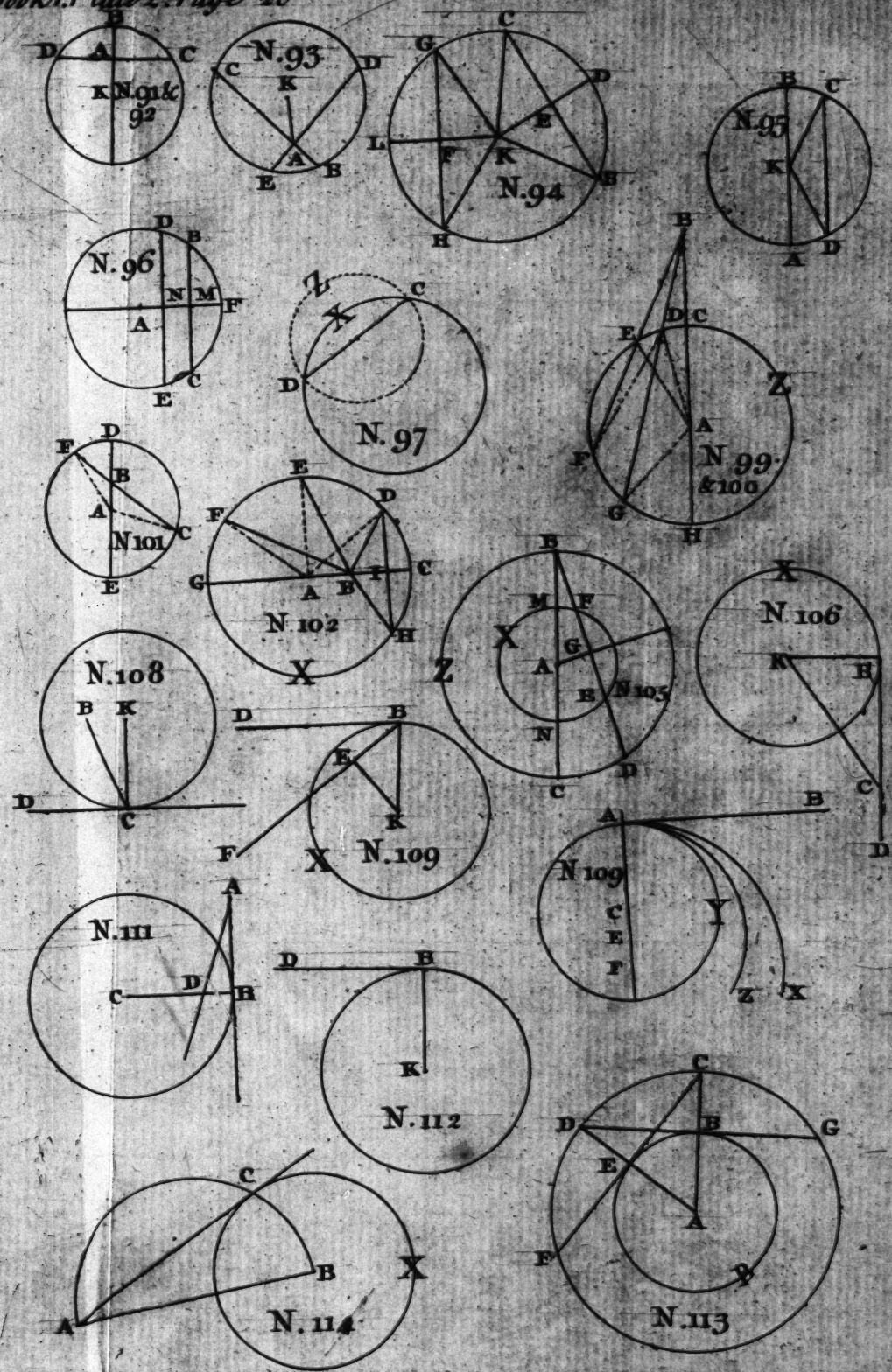
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B O O K. II.

OF Breadth, which is the second Species of EXTENSION.
Of Plane Surfaces.

S E C T. I.

OF Angles, or Surfaces which are between two Lines,
that meet indirectly.

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IN treating of Surfaces, we shall here consider only Planes;
that is to say, such Surfaces as are the shortest between two
right Lines; and shall begin first with those which are com-
prehended between two Lines, that meet or intersect in a Point.

DEFINITION 1. AN Angle is the aperture or opening of two
Lines, which meet indirectly.

TWO Lines which meet directly, degenerate into only one
Line.

DEF. 2. THE Point where two Lines meet which form an Angle,
is called the Vertex thereof, as is the Point B, and the two Lines are
called the Sides, or Legs of the Angle.

THUS, AB and BC are the Sides or Legs of the Angle ABC,
when an Angle is marked with three Letters as ABC, the middle-
most B denotes the Vertex, and the other two A and C, the
Sides.

DEF. 3. AN Angle which is upon a Plane, is called a plane Angle.

DEF. 4. THERE are three Sorts of Angles, so esteemed by reason
of their Sides, the Rectilinear, Curvilinear, and Mixtilinear, the
Rectilinear, is formed by the meeting of two right Lines, as C.

The Curvilinear, is formed by the meeting of two Curve Lines,
as A. The Mixtilinear, is formed by the meeting of a right
Line and a Curve, as B.

EVIDENT Propositions concerning Angles.

PROPOSITION 1. FROM the Definition of an Angle, it is evident,
that its Bigness does not depend upon the Length of the Lines which
form it, but upon their opening, or distance from each other.

FOR whether the Lines AB and AC are long or short, the
opening is always the same, and the Space, or Surface which is
between

between them, that is to say, which is nearest the Vertex A, will be neither augmented or diminished.

6 PROP. 2. *AN Angle cannot be either augmented or diminished, except one of its Sides in turning about the Vertex as upon a Center, either declines from, or inclines to, the other Side.*

7 PROP. 3. *ONE of the Sides of an Angle in turning round, and declining from the other Side, always makes the Angle greater, untill it becomes a right Line with the other Side, and then the Angle vanishes.*

8 PROP. 4. *ONE of the Sides of an Angle in turning about, can describe only one whole Round or Circle, after which, it joins to the other Side, and makes therewith only one Line.*

9 PROP. 5. *THE Arcs or Portions of different Circles, described through different Points of the Sides of an Angle, contain an equal Number of Degrees.*

THE Point X of the Side AB or AC, describes the Circle XX, and the Point Z the Circle ZZ, I say, that the Portions of the two Circles contained between the Radii AB and AC, are of an equal Number of Degrees. For since that they are described in the same Time, if we divide the Time into 360 Moments, the Circle having so many Degrees, in the first Moment, whilst Z describes the 360th Part of its Whole, it is evident, that X will also pass through the same Part of the Circle which it describes, for as the two whole Circles are completed in the same Time, each Part is likewise made proportionably.

10 PROP. 6. *A Circle being drawn from the Vertex of an Angle as a Center, the Portion of the Circle, contained between the Sides of the Angle, is the measure or content thereof.*

11 PROP. 7. *THE greatest Angle that is, cannot have all the Semi-circumference for its measure.*

FOR when the Leg AC of the Angle BAC, hath been turned about so as to make the Semi-circumference BCD, and that C is come to the Point D, it then makes only a right Line with AB: For since it is supposed, that BCD is the Semi-circumference, it must be, that AD+AB is the Diameter of the Circle, and that therefore, AD and AB are only one right Line, which divides the Circle into two equal Parts, the whole Circumference of the Circle is 360 Degrees, and consequently, the Semi-circumference is 180 Degrees; therefore an Angle cannot contain 180 Degrees, for when the Side AB is come to D, AC and AB make only one right Line.

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IF AC passeth beyond DA and goes to E, it will by this Means, form the Angle BAE greater than 180 Degrees; which some Geometers call a reverse Angle, and which is measured by the Arc BCDE, this Sort of Angle is not here considered, but only that of EAB, measured by the Arc EB, the Complement of the Arc BCDE to the whole Circle, to the Explanation of which we proceed.

OF the different Species of Angles with relation to their opening, or proportion to a Circle.

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THERE are three Sorts of Angles, with respect to their opening or proportion to a Circle, which are, the right Angle, the acute Angle, and the obtuse Angle.

DEF. 1. AN Angle that is measured by half the Semi-circumference, ¹² or fourth Part of the whole Circumference of the Circle, that is, an Arc of ninety Degrees, is called a right Angle.

THUS, supposing that the Arc AC is the fourth Part of the Circumference ACDE, and consequently ninety Degrees, which is the fourth Part of 360; which make the whole Circle, * the Angle ABC, which is measured by the Arc AC, is a right Angle.

DEF. 2. AN Angle that is measured by an Arc of more than ninety ¹³ Degrees, is said to be obtuse.

THE Angle FBC is obtuse, the Arc FC which measures it, being more than ninety Degrees; because it is greater than AC, the fourth Part of the Circle AFEDC.

DEF. 3. AN Angle which is measured by an Arc less than ninety ¹⁴ Degrees, is called Acute.

THE Angle FBE being measured by the Arc FE, which is less than the Arc AE of ninety Degrees, is Acute. The acute Angle may be diminished, *ad inf.* But the obtuse Angle can be augmented only 'till it approaches infinitely near to two right Angles.

DEF. 4. THE acute Angle, which together with the Obtuse is ¹⁴ equal to two right ones, is called the Complement of the obtuse Angle to a Semi-circle.

THUS, the Angle FBE, is the Complement of the obtuse Angle CBF.

THEOREM 1. A Line which is Perpendicular upon another Line, ¹⁶ makes therewith two right Angles; and if it makes therewith two right Angles, it is Perpendicular.

1. BA is Perpendicular upon A the Middle of CD, from which as a Center, having described the Semi-circle CBD, according to the Definition of a Perpendicular, the Lines or Chords BC and BD, are equal, * therefore the Arcs which they subtend, are equal, † and therefore since CBD is half of the Circumference, the Diameter of the Circle being CD, the Arcs BC and BD are Quadrants; then the Angles BAC and BAD are right ones, each being measured by the fourth Part of the Circle. ¶ 2. It is easy to demonstrate the second Part hereof; for if the two Angles CAB and DAB are right ones, the Arcs BC and BD are equal, each half the Semi-circumference; || but A and B, being therefore equally distant from C and D, the Line AB is Perpendicular. ‡

THEOREM 2. EVERY Line let fall upon another Line, makes therewith two Angles, equal to two right Angles. Eucl. 1. Prop. 13. ¹⁷

LET

* B. 1. n. 27.

* B. 1. n. 50. † B. 1. n. 31. ¶ Sup. n. 12. || Sup. n. 12.

‡ B. 1. n. 43.

Let the Line be EA, which falls upon the Line DC, I say, that the Angle DAE together with the Angle EAC, is equal to two right Angles: For having from the Point A as a Center, described the Semi-circle DEC, and raised the Perpendicular AB, whether the Angle DAE be right, obtuse, or acute, it is (according to the Definitions ¶) measured by the Arc DE, and the Angle EAC by the Arc EC,† these two Arcs together make the Semi-circumference of the Circle, which is equal to 180 Degrees, or two right Angles. §

18 COROLLARY. 1. It is evident, that how many Lines soever there be, which fall upon another Line at the same Point, they will form Angles, which altogether are equal to only two right ones.

19 COROLLARY. 2. THEREFORE, two or more Lines, which intersect at a Point, being prolonged will form Angles, which altogether are equal to only four right Angles.

Let as many Lines you please, be conceived to fall upon CD at the Point A, having from the Point A as a Center described a Circle, the Angles will be all measured by the Semi-circumference CGBD, which is the measure of two right Angles, having prolonged the Sides AB and AG, the Angles which they make are also equal to two right Angles, therefore all the Angles which can be made about the Point A, are equal to four right Angles.

20 THEOREM. 3. IF two right Lines meet at a Point of any right Line, making on both sides thereof, two Angles, equal to two right Angles, the two Lines meet directly. Eucl. 1. Prop. 14.

THE two Lines AB and AC, meet at the Point A of the right Line AE, and make on both sides that Line, the two Angles EAB and EAC, equal to two right ones. It must be proved, that they meet directly, that is, being taken together they make only one right Line. $BAE + CAE$ are equal to two right Angles. If it is said that BA and AC make two Lines, and that BA being prolonged goes to D, then by the preceeding Theorem, the Angles BAE and EAD, are equal to two right Angles; then $EAD = EAC$, which is absurd.

21 THEOREM 4. THE Vertical Angles of two Lines which intersect each other, are equal. Eucl. 1. Prop. 15.

THE two Lines BC and ED intersect at the Point A. I say, that the Angles DAC and BAE are equal, as also DAB and CAE, BAD and DAC are equal to two right Angles; DAB and EAB make also two right Angles: ¶ Therefore $BAD + DAC = BAD + EAB$, taking away from the two equal Quantities, the common Angle BAD, the remainders EAD and DAC are equal, the same reasoning will prove that $BAD = EAC$.

22 DEF. 4. THE Sine of an Arc is equal to half the Chord of twice that Arc, or to the Perpendicular let fall from the End of that Arc upon Radius.

THE

¶ Sup. n. 12, 13, and 14.

† Sup. n. 10.

¶ Sup. n. 17.

§ Sup. n. 12.

THE Arc EB is double the Arc BD, the Line BC, the half of BE, the Chord of BDE, is the Sine of the Arc BD, and also of the Arc BF, its complement to a Semi-circle, therefore the Arcs DE and BE, which together are equal to a Semi-circle, have the same Sine, and are complements to a Semi-circle, reciprocally each to each.

DEF. 5. THE Sine of an Angle, is the Sine of the Arc that measures it. 23

THUS BC which is the Sine of the Arc BD, the Measure of the Angle BAD, is the Sine of that Angle. When an Angle is obtuse, its Sine is also the Sine of the Acute Angle, which is its complement to a Semi-circle, thus BC is both the Sine of the obtuse Angle BAF, and of the acute Angle BAD, but 'tis only the Sines of the acute Angles, that are considered in the following Work, except the other are expressly mentioned.

BC being the Sine of the Angle BAD, the Line CD, contained between BC and the Arc BD, is called the versed Sine of that Arc BD, therefore BC is called the right Sine, or simply the Sine, CD being always distinguished by the Name versed Sine, the Line DE which touches the Arc ED, and which is bounded by AE and AD, which comprehend the same Arc, it called the Tangent of both that Arc, and also of the Angle BAD, the Line AE is called Secant, both of that Angle, and also of that Arc BD.

THEOR. 5. EQUAL Angles have equal Sines, and if the Sines are equal, the Angles are equal. 24

1. THE Angles GEF and CAB are equal, therefore having from A and E their Vertex, with an equal Extent described the Arcs CB and GF, which measure these Angles, these Arcs are equal,* continue them so that $GF=FN$, and $CB=BM$, then since that equal Arcs have equal Chords,† $CM=GN$, and consequently CO and GP, the Halves of these Chords, are equal, now these Halves are the Sines of the Angles CAB and GEF, according to the Definition, ‡ therefore the Sines of those Angles are equal. 2. If the Sines CO and GP are equal, $CM=GN$,§ and therefore $CBM=GFN$, then the Angles CAB and GBF, being measured by the Halves of those equal Arcs, they are equal.

THEOR. 6. IF a Line cuts two parallel Lines obliquely, it forms eight Angles, four of which, are alternate interior ones, and four Exterior. I say, the Alternates, either Interior, or Exterior, are equal. Ecul. 1. Prop. 29. 25

It must be proved, that $AFE=HGD$, and $AFG=FGD$, the two first, are alternate Exterior, and the two others, alternate Interior. Between the two Parallels, draw the Perpendiculars FK and GI, which are equal, || from F and G as Centers, with the distance GF, draw the Arcs AG and DF, which measure the Angles AFG and FGD, the Sines of these Arcs, or Angles, are the

* Sup. n. 10. † b. 1. n. 31. ‡ Sup. n. 22.
§ Sup. n. 22. || b. 1. n. 65.

the equal Perpendiculars GI and FK; * these Angles then are equal. † and AFE and AFG make two right Angles, ‡ as do FGD and DGH, therefore $AFE + AFG = FGD + DGH$, taking from each part, the equal Angles AFG and FGD, there will remain $AFE = DGH$.

26 THEOREM. 7. If a Line which joins two other Lines, makes the Alternate Angles equal, the two Lines are Parallel. Eucl. 1. Prop. 27.

If AFG and FGD are equal; their Sines GI and FK are equal, § GI is Perpendicular upon AB, and FK upon CD, therefore by the Definition of Sines, || the two Lines AB and CD, * are Parallel.

27 THEOREM. 8. If a Line cuts two or more Parallel Lines, all the Angles that it makes with them taken on the same Side, are equal.

It is to be proved, that $GAY = ABX = BCZ$, that $YBC = ZCH$, and that on the other Side it is the same; that $GAF = ABE = BCI$, and $FAB = EBC = ICH$. 1. $ZCB = CBE$, * and $CBE = ABX$, † then $ZCB = CBE = XBA$; therefore according to this Axiom, that two Quantities equal to a Third are equal. $ZCB = XBA$; In the same Manner it is demonstrated, that $GAY = ABX$ &c.

28 THEOREM. 9. If a Line that falls upon several other Lines, makes therewith, the Angles taken on the same Side, equal, the Lines are Parallel. Eucl. 1. Prop. 28.

$GAY + YAB$ is equal to two right Angles, as is also $GBX + GBE$, † then taking away from the two equal wholes, the Angles GAY and GBX , which are supposed equal, the Alternate Angles YAB and GBE , will remain equal. Therefore the Lines Y and Z are Parallel, § It is demonstrated in the same Manner, that Y and Z, or X and Z are Parallel.

29 PROBLEM. 1. FROM a Point given in a right Line, to describe a Rectilinear Angle, equal to an Angle given. Eucl. 1. Prop. 2.

FROM the given Point A of the right Line Z, it is required to describe an Angle CAB, equal to the Angle EDF, from the Point D as a Center, draw the Arc EF, then from the given Point A as a Center, with the extent of DE, draw the Arc X, from which take the Arc BC, equal to the Arc EF, by means of equal Chords EF, BC, || then drawing a right Line from C to the Point A, the Angle CAB, will the Angle required, equal to EDF, for they are measured by equal Arcs, therefore they are equal *.

30 PROBLEM. 2. FROM a Point given out of a Line, to draw a right Line upon another Line, so as to make therewith an Angle, equal to an Angle given.

LET

* Sup. n. 23. † Sup. n. 24. ‡ Sup. n. 17.

§ Sup. n. 24. || Sup. n. 22.

* b. 1. n. 71. † Sup. n. 25. ‡ Sup. n. 21.

§ Sup. n. 17, || Sup. n. 26.

* b. 1. n. 3. * Sup. n. 10.

LET the given Point be D, from which must be drawn upon Z, a right Line making therewith an Angle, equal to X. From any Point of Z taken at Pleasure, draw a Line by the preceeding Problem, making therewith the Angle CAB, equal to the given Angle X, such is AC, if this Line pass thro' the Point D, the Problem is done. If it doth not pass thro' D, draw a Line thro' D, parallel to CA, † then $DEB = CAB = X$, ‡ therefore $DEB = X$.

PROBLEM. 3. To bisect, or divide a given Angle, into two equal Parts. Eucl. 1. Prop. 9. ³¹

THE given Angle is BAC; having made the two Sides AB and AC, equal, join them by the Line BC, upon which having drawn a Perpendicular AG, § from the Point A. $BD = DC$, || therefore the Arc BGC, being conceived as part of a Circle whose Center is A. AB and AC the Radii thereof, and BC the Chord which is cut equally in two by the said Perpendicular AG, as is also the Arc BC at the Point G. * therefore the Arc BG is equal to the Arc GC, and therefore the Angles BAG, GAC which they measure are equal, † the Angle BAC is then cut equally in two.

THEOREM. 10. THE Angle comprehended between a Circle and its Tangent, is lesser than any rectilinear Angle whatsoever, Eucl. 3. ³² Prop. 16.

BETWEEN the Circle and its Tangent cannot be drawn a right Line, † therefore the mixt Angle made by the Circle and its Tangent, cannot be divided; this Angle is therefore lesser than any rectilinear Angle whatsoever.

SECTION. II.

OF the Comparison of Angles, and of their different Position with respect to a Circle.

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AN Angle, as has been aforesaid, is measured by the Arc of a Circle whose Center is the Vertex of the said Angle, and which is bounded by the Sides which form it. Now the Vertex of an Angle may be either at the Circle's Center, or out of the Center, in the Circumference, or out of the Circumference; and altho' the Angle may in all these cases be the same, yet it affords different considerations.

DEFINITION. 1. THE Angle whose vertex is in the Circumference of the Circle, and the Sides in the Circle, is called an Angle at the Circumference, as BAC.

F

Euclid

† b. 1. n. 72.

‡ Sup. n. 27.

§ b. 1. n. 46.

|| b. 1. n. 42.

* b. 1. n. 84.

† Sup. n. 10.

‡ b. 1. n. 109.

EUCLID calls those inscribed Angles, which are in the Circumference; and those Circumscribing ones which are out of the Circle, but which touch it with their Sides; BAC is an inscribed Angle, and EDF a Circumscribing one.

34 DEF. 2. An Angle at the Center, is an Angle whose Vertex is at the Center of the Circle, and whose sides are Radii's, as O M N.

35 DEF. 3. A Segment of a Circle, is that part of the Circle which is included between the Circumference, and a Chord Line, thus PR divides the Circle into a greater, and a lesser Segment.

36 DEF. 4. THE Angle made by a Tangent and a Chord Line, or by a Secant, drawn from the Point of contact, is called an Angle of the Segment, as $\angle P R$ or $R P T$.

37 THEOREM. 1. THE Angle of the Segment is measured by half the Arc that is bounded by its Chord. Eucl. 3. Prop. 32.

LET the Angle be CBE, made by the Tangent CB, and the Chord BE, it must be proved, that its measure is half the Arc BE. Let there be drawn the Diameter GH cutting the Arc and the Chord EB into two equal Parts; and draw also the Diameter DF Parallel to the Chord EB, and the Radius AB passing thro' the Point of contact, will form an Angle at the Center GAB, which is measured by the Arc GB. It remains then only to prove that the Angle CBE is equal thereto, which is evident; for the Angle CBA is a right one as well as the Angle GAF; * but the Angle EBA is equal to the Angle BAF, because they are Alternate ones, † then taking away the two equal Angles, viz. FAB from GAE which is a right Angle, and ABE from the right Angle ABC, the Remainders BAG and CBE are equal. Now BAG is measured by the Arc BG; then CBE which is equal thereto, is measured by an Arc equal to BG, the half of BGE; which was to be proved. The same Way of reasoning will prove, that the other Angle DBE is measured by the Arc BH, the half of the Arc BHE, comprehended in the said Angle between the Tangent BD and the Chord BE, by adding to the right Angles DBA, FAH, the equal alternate Angles EBA, BAF, instead of Subtracting as in the preceeding Demonstration.

38 COROLLARY. IT is evident, that if an infinite Number of Arcs be drawn between the Tangent and the Chord, and which are crossed by producing the Chord, they are all of an equal Number of Degrees, seeing they are the Measure of one and the same Angle.

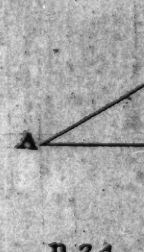
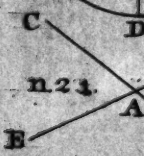
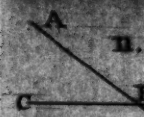
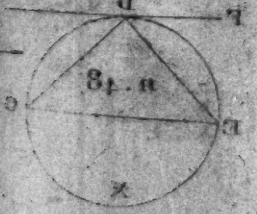
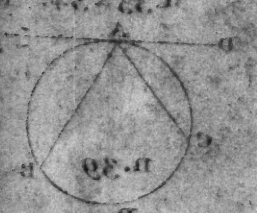
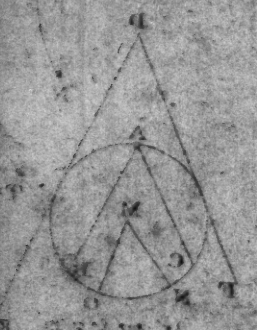
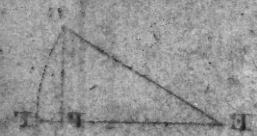
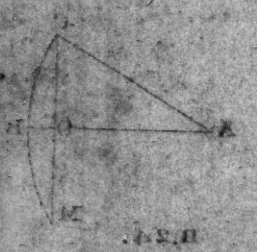
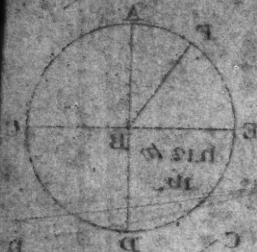
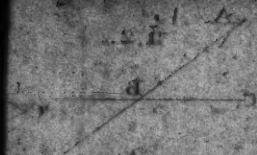
FOR the Angle BAC being measured by the half of AD, of AE, of AF, &c. If this Angle for Example contain'd ten Degrees, all these Arcs will be each twenty Degrees.

39 THEOREM. 2. THE Angle at the Circumference, is measured by half the Arc which it stands upon.

LET the Angle be BAC. It must be proved that it is measured by half the Arc BC. Thro' the Point A let there be drawn the

* b. 1. n. 91 & 103.

† Sup. n. 25.



was the Center, this would be evident. As the Point G is the Center, thro' G draw Parallels to the Sides of the Angle CBD, the Angle CBD is equal to the Angle IGH, * whose measure is the Arc HI. It is then required only to prove, that the Arc HI is half the Arcs DC and AE. The Arcs HI and FL are equal. † $DI = AL$ and $CH = EF$. ‡ Now $HI = AE + AL + EF$. As also $HI = DC - CH$ (or EF) $- DI$ (or AL), that is to say, $HI = DC - EF - AL$. Consequently $HI + HI = AE + AL + EF + DC - EF - AL$. Now $+AL + EF - AL - EF =$ Cypher, then $HI + HI$ (or $2 HI$) $= AE + DC$. Consequently the Arc HI is also half the Arcs DC and AE; which are therefore the Measure of the Angle CBD equal to IGH, whose measure is the Arc HI.

§ 3 THEOREM. 5. *AN Angle whose Vertex is without the Circle, but whose Sides cross it, and stand upon the Circumference, is measured by half the concave Arc less half the convex Arc.*

LET the Angle be BAC. I say that it is measured by half the Arc BC less half the Arc DF. Thro' D draw a Line parallel to AC; § therefore the Angle BAC is equal to BDE, || which is measured by half the Arc BE. * The Arcs CE and DF are equal. † Now $BE = BC - CE$; then $BE = BC - DF$. Then the half of BE is equal to half $BC - DF$, therefore since the Angle BAC is measured by half BE; its measure is half the concave Arc BC, upon which it stands, less half the convex Arc DF; which was to be proved.

§ 4 THEOREM. 6. *AN Angle whose Sides touch the Circle, is measured by half the concave Arc less half the convex Arc.*

LET the Angle be BAC, whose sides AB, AC, touch the Circle. It is to be proved, that its measure is half BEC less half BFC. Thro' C one of the Points of contact, draw CE parallel to AB, ‡ the Angle BAC is equal to the Angle DCE. § Now DCE is measured by half the Arc CE, || which also measures CAB: It remains therefore to prove, that the Arc $BEC - BFC$ is equal to the Arc CE. 1. The Arc $BEC - BE$ is equal to the Arc EC. 2. Since the Arc BC * is equal to the Arc BE; then $BEC - BFC = EC$: which was to be proved.

* Sup. n. 27. † Sup. n. 21. and n. 42. ‡ b. 1. n. 86.

§ b. 1. n. 72. || Sup. n. 27.

* Sup. n. 39. † b. 1. n. 86. ‡ b. 1. n. 72.

§ Sup. n. 27. || Sup. n. 37.

* b. 1. n. 36.

S E C T. III.

OF TRIANGLES

DEFINITION. I.

*A Triangle is a Surface bounded by three Lines. There are fix 55
Sorts thereof: Three in respect of their Sides, and three with
regard to their Angles.*

DEF. 2. *A Triangle whose three Sides are equal is called Equi- 56
lateral, as ABC.*

DEF. 3. *A Triangle that has only two Sides equal is called 57
Isocles, as HGI.*

DEF. 4. *A Triangle whose Sides are all unequal is called Sca- 58
lene, as DEF.*

DEF. 5. *A Triangle which hath one right Angle, is called a 59
Rectangled Triangle, as LKM.*

DEF. 6. *A Triangle that hath one obtuse Angle, is called Ob- 60
tuse-angled, or Amblygenium, as NPO.*

DEF. 7. *A Triangle whose Angles are all Acute, is called 61
Acute angled, or Oxygenium, as ABC.*

DEF. 8. *A Triangle is said to be inscribed in a Circle, when 62
its three Angular Points are in the Circumference thereof, and the
Circle is then said to circumscribe the Triangle.*

DEF. 9. *A Triangle is said to circumscribe a Circle, when all 63
its Angular Points are without the Circle, and whose Sides touch the
Circle; and then the Circle is said to be inscribed in the Triangle.*

DEF. 10. *Two Triangles are Equiangular, when the Angles of 64
one are equal to the Angles of the other, each to each.*

DEF. 11. *Two Triangles are said to be entirely equal, which 65
being Equiangular, and the Sides containing the Angles of one equal
to the Sides containing the Angles of the other, each to each.*

THEOREM 1. *ANY two Sides of a Triangle, are together longer 66
than the third. Eucl. 1. Prop. 20.*

*AB+BC is longer than AC; It has been already asserted, that
between the two Points A and C, there cannot be any Line con-
ceived shorter than the right Line AC. **

COROLLARY. *THEREFORE a Triangle cannot be made of three 67
given Lines, unless two of them are together longer than the third,
Eucl. 1. Prop. 22.*

PROBLEM. I. *THREE Lines being given, to make thereof a 68
Triangle, Eucl. 1. Prop. 1. and 22.*

*LET the three given Lines be A, B, C. from the Ends of one
of the three Lines, as C for Example, with the Extent of the
Line*

* b. 1. n. 12.

Line A describe an Arc, and from the other End with the Extent of B draw another Arc; then having drawn from the Point where these two Arcs intersect, the two Lines A and B to the Ends of C, they will make the Triangle required, whose Sides are by construction equal to the given Lines.

69 PROBLEM. 2. To describe an equilateral Triangle upon a Line given.

LET the given Line be AB, with the extent of this Line, and from the Ends A and B as Centers, describe two Arcs, from the Point where they intersect, draw Lines thro' A and B, and they will make an Equilateral Triangle.

70 PROBLEM. 3. To circumscribe a Circle about a given Triangle. Eucl. 4. Prop. 5.

Is the same thing as to make a Circle pass thro' three given Points.*

71 COROLLARY. It is evident that to find a Point equally distant from three given Points, which are not upon the same Line, a Circle must be described thro' those three Points; the Center of that Circle is the Point required.

72 DEF. 12. In every Triangle, the Angle formed by producing one of its Sides, is called an exterior Angle.

In the Triangle BAC, having produced BC to D, the Angle ACD is called an exterior Angle: being considered as opposite to the two Interior ones CAB and ABC.

73 THEOREM. 2. The exterior Angle of a Triangle is equal to the two opposite Interior ones. Eucl. 1. Prop. 32.

Draw CE parallel to AB, and the exterior Angle ACD is thereby divided into two Angles ACE and ECD, which must be proved equal to the two opposite Interior ones, which is evident; for $ABC = ECD$, † and $BAC = ACE$. †

74 COROLLARY. The exterior Angle is therefore greater than either of the opposite Interior ones. Eucl. 1. Prop. 16.

THIS must certainly be so, by reason the exterior Angle is equal to both the Interior opposite ones, as has been demonstrated.

75 THEOREM. 3. The three Angles of a Triangle are together equal to two right Angles. Eucl. 1. Prop. 32.

To demonstrate which, circumscribe a Circle about the given Triangle. § The three Angles of the Triangle are measured by half the Arcs which they subtend, || therefore they are measured by half the Circumference of the Circle, that is 180 Degrees, the Sum of two right Angles. *

OR THUS. The two Angles ACB and ACD are equal to two right Angles; † but ACD the exterior Angle, is equal to the two opposite Interior ones ABC and CAB, ‡ therefore these two Angles together with ABC are equal to two right Angles.

COROL-

* b. 1. n. 87. † Sup. n. 27. ‡ Sup. n. 25.

§ Sup. n. 70. || Sup. n. 39.

* Sup. n. 12. † Sup. n. 17. ‡ Sup. n. 73.

COROLLARY. 1. THEREFORE the two Angles of a Triangle being known, the third is also known. 76

FOR the three together are equal to 180 Degrees, if two of them make 160, the third ought to make 20.

COROLLARY. 2. THE three Angles of a Triangle may be all Acute. 77

FOR one hundred and eighty Degrees, which is the Sum of the three Angles of a Triangle, may be divided into three parts, and each of them will be less than the Sum of either a right, or an obtuse Angle, and which all together make only one hundred and eighty Degrees, the Sum of two right Angles.

COROLLARY. 3. IN a Triangle there can be but one right Angle, nor but one obtuse Angle. 78

IF there were two right Angles, the three would together make more than two right Angles. If two were obtuse, the obtuse Angle being greater than the right Angle, the three together would also be more than two right ones; which is contrary to the preceding Theorem.

COROLLARY. 4. TWO of the Angles of a Triangle must therefore be Acute, or less than two right Angles. Eucl. 1. Prop. 17. 79

THIS is evident, since that a Triangle cannot have two of its Angles, either Right, or Obtuse.

COROLLARY. 5. IF in two Triangles, two Angles of one are equal to two Angles of the other, they are equiangular, that is, the third Angle of the one is equal to the third Angle of the other. 80

COROLLARY. 6. IF a Triangle hath one of its Angles greater than that of another Triangle, the other two are together less than the other two of the said Triangle. 81

FOR a greater part being subtracted from the Sum of two right Angles, the remainder ought to be lesser, than it would if a lesser part had been taken away.

THEOREM. 4. THE three Angles of a Scalene Triangle are unequal. 82

LET ABC be a Scalene Triangle, circumscribe it with the Circle X. * Since that the three Sides AB, AC, BC, are unequal, the three Arcs of which they are the Chords are unequal, † Consequently the three Angles of the Triangle ABC, which are measured by half these Arcs, ‡ are unequal.

IN an Isosceles Triangle, the Angles at the Base are equal; and if the Angles at the Base are equal; the Triangle is Isosceles. Eucl. 1. Prop. 5. 83

LET ABC be an Isosceles Triangle, circumscribe it with the Circle X, since that $AB=AC$, the Arcs subtended by those Sides are equal. Now the Halves of these Arcs are the Measure of the Angles ABC and ACB; § these Angles are therefore equal. The other part of this Proposition is easy. For if the two Angles

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ABC

* Sup. n. 70. † b. 1. n. 32. ‡ Sup. n. 39.
§ Sup. n. 39.

ABC and ACB are equal, the Arcs AB and AC, or their Chords are equal; therefore the Triangle ABC is an Isosceles.

84 COROLLARY. 1. THE Angles at the Base of an Isosceles Triangle, can neither of them be either Right, or Obtuse.

FOR if one be a Right one, the other would be the same also: thus the Sum of the three Angles would be more than two right Angles; which cannot be. * and if one was obtuse, the other would be so likewise; so that only two Angles of the Triangle would make more than two right ones: which is also impossible.

85 COROLLARY 2. If either of the Vertical Angles, or one of the two Angles at the Base of an Isosceles are Equal, their Angles are all equal.

FOR if the equal Angle be at the Base, the other Angle at the Base will be also equal, and so likewise the third, † 2. If the equal Angle be at the Vertex, the Sum of the two Angles at the Base of each Triangle is the same; now each is half that same Sum, therefore they are equal.

86 THEOREM. 6. THE three Angles of an Equilateral Triangle are equal.

HAVING describ'd a Circle about the Equilateral Triangle, the Sides of the Triangle being the Chords thereof, are consequently equal, † and their halves equal. § Now these halves measure the Angles of the Triangle. Therefore the Angles are all equal.

87 COROLLARY. HENCE each Angle of an Equilateral Triangle is Acute, and always of sixty Degrees.

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88 FROM this Corollary is deduced a different Method, for raising a Perpendicular from a given Point of a Line, for Example, from the Point A of the Line AB; make the Equilateral Triangle ABC, then produce BC to D, so that $CD = CB$; from D to A draw a right Line, which will be the Perpendicular, if BAD is a right Angle. Now I demonstrate that it is; for the Angle ACB is equal to the two opposite ones D and A which are equal, by reason CD is equal to CA. Hence ACB being 60 Degrees, DAC is 30, CAB is 60; therefore the Angle BAD is $60 + 30$, or 90 Degrees; which is the content of a right Angle.

89 THEOREM. 7. IN a Triangle, the longest Side subtends the greatest Angle, and the greatest Angle is subtended by the longest Side. Eucl. 1. Prop. 18. and 19.

HAVING described a Circle about the Triangle, the longest Side of the Triangle will subtend the greatest Arc. || Now the half of that Arc is the Measure of the Angle opposite to the longest

* Sup. n. 75.

† Sup. n. 80.

‡ b. 1. n. 31.

§ Sup. n. 39.

|| b. 1. 32.

longest Side,* therefore that Angle is the greatest, which is measured by half the greatest Arc. If a Triangle be inscribed in a Circle, the greatest Angle is measured by half the greatest Arc, now the greatest Arc has the longest Chord;† hence the Side opposite to that Angle is the longest.

THEOREM 8. IF two of the Angles of a Triangle are equal, the Sides opposite to those Angles are equal. Eucl. 1. Prop. 6.

HAVING inscribed the Triangle in a Circle, the equal Angles are subtended by equal Arcs, having equal Chords, which are the Sides opposite to those Angles.

THEOREM 9. IN a Triangle, the half of each Side is the Sine of the opposite Angle,

LET the Triangle ABC be inscribed in the Circle X: It must be demonstrated that AD, the half of AC, is the Sine of the Angle ABC. The Arc AE, which is the half of the Arc AC, measures the Angle ABC;* hence the Arc AC is double the Arc which measures the Angle ABC; therefore AD, half the Chord of the Arc AC, is the Sine of the Arc AE,† and also that of the Angle ABC. The same way it is demonstrated, that half BA is the Sine of the Angle ACB, and the half of BC that of BAC.

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THEREFORE the Sine of one Angle, is to the Side opposite to that Angle, as the Sine of another Angle, is to its opposite Side: Or the Sines of Angles, are to each other as the opposite Sides, seeing that the Halves are as the Wholes.

THEOREM 10. TWO Triangles whose Sides are equal, are Equi-angular, and entirely equal to each other.

THE Sides of the Triangles ABC and DEF are equal, I say, that they are Equi-angular, and entirely equal, or (which is the same thing) that being placed one upon another, they will coincide. 1. BC being equal to DE, it is plain, that if the Line DE be placed upon BC, they will coincide together. If it is said that DF will not coincide with AB, nor FE with AC, I prove the contrary. From B as a Center, with the distance of the equal Lines AB or DF, describe the Arc Z; and from C with the extent of AC or EF (which are equal,) draw the Arc X; these two Arcs must necessarily intersect as here at A. 2. It is evident that D being placed upon B, the Point F must necessarily be found in the Arc Z, and that E being placed upon C, the Point F, will be found in Z and X, and therefore in the Point A, where the two Circles intersect: Hence, the two Triangles being placed one upon the other, will every where agree; and are equal. Q. E. D.

IT may be said that the two Arcs Z and X, will cross in another Place beside A, which is true, but by what has been demonstrated, it can be only in two Points,‡ and the second Point must

G 2

be

* Sup. n. 39.

† B. 1. n. 32.

* Sup. n. 39.

† Sup. 22 and 23.

‡ B. 1. n. 89.

be below BC (*viz.*) at the Point G, as is evident; hence, if they cross each other above BC, in any other Point beside A; they would intersect in three Points; which cannot be.*

- 93 COROLLARY 1. If from the Ends of a right Line two other right Lines are drawn which meet at a Point, there cannot be drawn two other Lines from the same Ends, and on the same Side, equal to the two first, each to each, so as to meet in another Point. Eucl. 1. Prop. 7.

For they make two Triangles having equal Sides, which consequently being Equi-angular, ought to coincide together.

- 94 COROLLARY 2. Two Triangles having two Sides equal, each to each, and the Bases equal, the Angles comprehended by the two equal Sides, are equal. Eucl. 1. Prop. 8.

The two Triangles having their Sides equal, are entirely Equi-angular.

- 95 THEOREM 11. Two Equi-angular Triangles, which have one Side equal, are entirely equal.

ABC and DEF are Equi-angular, and $AB = DE$, I say, they are entirely equal, for if one is placed upon the other, DE will coincide with AB, since they are equal Lines. If DF does not coincide with AC, but with AG, as the Angles FDE and CAB are equal, it will follow that the Angle $CAB = GAB$, which is the same as FDE; which cannot be, it must therefore be that DF will coincide with AC, and by the same reason FE with BC, and the Point F with C, and consequently the two Triangles are both every way equal.

- 96 COROLLARY. Two Triangles having two Angles and one Side equal, are entirely equal. Eucl. 1. Prop. 26.

For two Triangles which have two Angles equal, are entirely Equi-angular,* consequently if they have one Side equal, it must be, according to the Theorem, that they are entirely equal.

- 97 PROBLEM 4. To make a Triangle, having one Side, and the Angles at that Side given.

The given Side is AB, let the Angle at the Point A be called X, and that at B, be called Z. Upon A draw the Line AC, making the Angle CAB equal to X,† and upon B draw the Line BD, making the Angle ABD equal to Z; these two Lines will intersect, unless X and Z are either right Angles, or both together greater than two right Angles, for if they are, the Problem would be impossible,‡ and the two Lines would not intersect, but be parallel,§ or else decline from each other, and they must incline in order to make a Triangle such as ABE, which is that which was proposed to be made, for 1. Having two equal Angles, they are Equi-angular; || and having one Side equal, they are entirely equal; by the preceeding Corollary.

- 98 THEOREM 12. Two Triangles that have one Angle equal, and the Sides containing that Angle equal, are wholly equal. Eucl. 1. Prop. 4.

THE

* B. 1. n. 88.

* Sup. n. 80.

† Sup. n. 29.

‡ Sup. n. 79.

§ B. 1. n. 68.

|| Sup. n. 80.

THE Angle $BAC = EDF$, and $AB = DE$, and $AC = DF$, I say, that the two Triangles being put one upon another will coincide; For DE will coincide with AB ; if DF doth not agree with AC , but with AH , the Angle BAC will be equal to the Angle BAH , which cannot be: Therefore DF agrees with AC , and the Point F with C , consequently $BC = EF$; therefore the two Triangles having their Sides equal, are Equi-angular,* that is, equal in every Part.

THEOREM 13. TWO Triangles having one and the same Base, the Vertical Angle of the contained Triangle, is greater than the Vertical Angle of the containing Triangle. Eucl. 1. Prop. 21. 99

THE two Triangles CAD and CBD , have CD for their Base. CAD is contained in CBD , it is to be proved, that $A \searrow B$. Prolong DA to E , the exterior Angle CAD is equal to the two interior opposite ones AEC , ECA ; † therefore it is greater than either of them, by the same reason $CED = EBD$, BDE ; and is therefore also greater than either of them, consequently CAD being greater than CED , it is also greater than CBD ; which was to be proved.

PROBLEM 5. TO inscribe in a Circle, a Triangle Equi-angular to a given Triangle. Eucl. 4. Prop. 2. 100

LET the given Triangle be FED , which must be inscribed in the Circle BAC , draw the Tangent GH , ‡ and make the Angle GAB equal to FDE , and CAH equal to DFE , § compleat the Triangle ABC by drawing the Line BC . Since the Angle BAG , whose measure is half the Arc BA , † (which is also the measure of the Angle BCA) is equal to FDE , then $BCA = FDE$. By the same Reason CAH , (equal to DFE ,) having for its measure half the Arc AC , (the measure of CBA ,) it must be, that CBA and DFE are equal; hence the two Triangles ABC and FED , having two Angles equal, they are entirely Equi-angular, || having therefore inscribed the Triangle ABC , that which was proposed will be done.

PROBLEM 6. TO describe about a Circle, a Triangle Equi-angular to a given Triangle, or to circumscribe a Circle with a Triangle Equi-angular to a given Triangle. Eucl. 4. Prop. 3. 101

THE given Triangle is KLM , and the Circle is X , having drawn the Radius AD , make on one Side thereof the Angle $BAD = NLK$, † and on the other $DAC = KMH$; then having drawn through the three Points B , D , C , the Tangents EF , FG , and GE ; ¶ I say, that the Triangle EFG , is Equi-angular to KLM . The six Angles of the two Triangles BAD and BED , make four right Angles, † AB and AD being Perpendicular to each other, $EBD + ABD$ make a right Angle, also $EDB + BDA$ make a right Angle, therefore $BED + BAD$ make two right Angles; now by Construction, $BAD = KLN$, the Angle $KLN + KLM$

* Sup. n. 92.

† Sup. n. 73.

‡ B. 1. n. 112.

§ Sup. n. 29.

† Sup. n. 37

¶ Sup. n. 80.

‡ Sup. n. 29.

¶ B. 1. n. 121.

† Sup. n. 75.

KLM makes two right Angles, therefore $BED = KLM$. The same way it is demonstrated, that $DGC = KML$, and consequently that $EFG = LKM$, and that the Triangles EFG, and LKM, are therefore Equi-angular. §

102 PROBLEM 7. To inscribe a Circle in a Triangle. Eucl. 4. Prop. 4.

ACCORDING to what has been taught,* divide the Angles CBD, and CDB, into two equal Parts, by the Lines BA and DA, and from the Point A where these two Lines intersect; draw the Perpendiculars AE, AF, AG, upon the Sides of the Triangle, ¶ then from A, with the extent of one of these Lines, describe the Circle X, which will be inscribed in the Triangle BCD; to prove which, it must be shewn that the three Lines AE, AF, AG, are equal. The two Triangles AEB, and AFB are Rectangular by Construction, since that AE and AF were drawn perpendicularly; the Angles EBA and ABF are equal by Construction; hence these two Triangles having two Angles equal, they are Equi-angular;* the Side AB is common, therefore they are entirely equal. † Thus $AE = AF$; it is demonstrated the same way, that AG is equal to AF, and to AE; which was to be proved.

103 COROLLARY. THREE such Lines being given, as being prolonged will make a Triangle, there may be found a Point equally distant from each.

THE Triangle being made, inscribe a Circle therein, the Center thereof is the Point required.

104 THEOREM 14. TWO Triangles having two Sides of one, equal to two Sides of the other, but which contain unequal Angles, that which hath the greatest Angle, will also have the longest Base, and reciprocally, that which hath the longest Base, has the greatest Angle. Eucl. 1. Prop. 24 and 25.

LET the two Triangles be ABC, DEF, if $AB = DE$, and $BC = EF$, and the Angle $B > E$; I say, the Base $AC > DF$. 1. For if the Triangle ABC, be conceived as placed upon the Triangle DEF, so that the Points A and B coincide with D and E: As the Angle ABC was supposed greater than the Angle DEF, the Side BC, will not coincide with EF, but will be nearer EG, as here at EH, and the Line DH will be equal to AC,* then imagining the Line $EF = EH$ as turning round upon E, the End of DE, as it approaches to the other End EG, when it is at the Point H, the Line DH which will join their other Ends, will be longer than DF, † which was to be demonstrated in the first place. 2. The inverse Part hereof will be demonstrated in like Manner, (viz) That the Angle $DEH = ABC$, is greater than the Angle DEF, the Line AC or DH being supposed longer than DF, and the Line EH being supposed nearer than EF to the Line EG, when turned round upon the Point E, as before-mentioned. ‡

S E C T.

§ Sup. n. 80.

* Sup. n. 31.

¶ B. 1. n. 46.

* Sup. n. 80.

† Sup. n. 96.

* Sup. n. 89.

‡ B. 1. n. 100.

† B. 1. n. 100.

SECT. IV.

OF Figures having several Sides.

DEFINITION 1. **Q**UADRILATERALS, or Figures of four Sides, 105
have different Names.

1. IF the opposite Sides are parallel, 'tis a *Parallelogram*. 2. If the four Sides are equal, and all the Angles right ones, 'tis a *Square*, as A. 3. If the four Sides are equal, and also the opposite Angles equal, but not right ones, 'tis a *Rhombus*, or *Lozenge*, as B. 4. If the Sides are not all equal, but all the Angles are right ones; 'tis a *long Square*, *Oblong*, *rectangled Parallelogram*, or simply a *Rectangle*, as C. 5. If only the opposite Sides are equal, and the opposite Angles also equal, but not right ones, the Figure is called a *Rhomboides*, as D. 6. All other *Quadrilaterals*, whose opposite Sides are neither parallel nor equal, are called *Trapezias*, as E.

DEF. 2. A Figure is called *Regular*, when all its Sides and all its Angles are equal. 106

DEF. 3. A Figure of several Sides is in general called a *Polygon*, it takes the Name which is proper to it, from the Number of its Sides, or from the Number of the Angles which its Sides contain: Thus a Figure of five Sides is called a *Pentagon*; of six, a *Hexagon*; of seven, a *Heptagon*; of eight, an *Octagon*; of nine, a *Enneagon*; of ten, a *Decagon*; of eleven, a *Endecagon*; of twelve, a *Dodecagon*, &c. 107

DEF. 4. A regular Figure is said to be inscribed in a Circle, when all its angular Points are in the Circumference of the Circle; but when all its Sides touch the Circumference of the Circle, it is said to circumscribe the Circle. 108

DEF. 5. TWO Lines being drawn from the Ends of one of the Sides of a regular Figure, to the Center of the Circle in which it is inscribed, (or which it circumscribes,) make an Angle, which is called an Angle at the Center, and the Angles which the Sides make, taken two and two, are called Angles of the Figure. 109

LEMMA 1. OBLIQUE Lines which make equal Angles between the same Parallels, are equal. 110

LET there be drawn the Perpendiculars AC and DF between the Parallels Z and X, they are equal,* but the Angles ACB and DFE being right ones, and the Angles ABC and DEF equal by the Hypothesis, the two Triangles ABC and DEF are then Equiangular, † wherefore AC is equal to DF, they are all equal; ‡ consequently AB=DE. Q. E. D.

LEMMA 2. LINES which make the same Angles, upon one and the same Line, are parallel. 111

THE

* B. 1. n. 65.

† Sup. n. 80.

‡ Sup. n. 96.

THE Lines AB and DE make upon Z and X, the same Angles, hence $ABC = DEF$, consequently AB and DE ought to be parallel.*

112 LEMMA 3. TWO Lines which join two equal and parallel Lines, are equal. Eucl. 1. Prop. 33.

IF AD and BE, join the two equal parallel Lines AB and DE, I say, that they are also equal, the Perpendiculars AC and DF are parallel,† AB and DE being parallel, the Angles ABC and DEF are equal,‡ now ACB and DEF are right Angles; therefore the two Triangles ABC and DEF, being Equi-angular § and having one Side equal, since that $AB = DE$, they are entirely equal;* therefore $BC = EF$; then $CF = BE$: Now AD and CF which are between the Parallels AC and DF, are equal,† therefore $BE = CF = AD$; hence $BE = AD$.

113 THEOREM 1. THE four Angles of a Quadrilateral, are equal to four right Angles.

LET the Quadrilateral be ABCD, it is required to prove, that its four Angles are equal to four right Angles: the Line AC being drawn from any one Angle, to the opposite Angle, it will divide the Figure into the two Triangles ABC and ACD, the three Angles of each Triangle make two right Angles.‡ Hence all the Angles of ABCD, are equal to four right Angles.

114 THEOREM 2. THE opposite Angles of a Quadrilateral inscribed in a Circle, make two right Angles. Eucl. 3. Prop. 22.

THE Quadrilateral ABCD is inscribed in a Circle; the two opposite Angles ACD and ABD, are each measured by half the Arc upon which they are placed; § now they all together being placed upon the whole Circumference, are therefore measured by half the whole Circumference, are therefore equal to two right Angles.‖ It is the same of the other two Angles BAC and BDC.

115 COROLLARY. IF the opposite Angles of a Quadrilateral do not make two right Angles, it cannot be inscribed in a Circle.

FOR if it could, the opposite Angles of the Quadrilateral would be equal to two right ones by the Theorem, but by the Supposition they are not; which is impossible.

116 THEOREM 3. IF the opposite Sides of a Quadrilateral are equal; they are parallel.

$AB = CD$, and $BC = AD$, the Side BD is common; then the two Triangles ABD and BCD being equal, they are Equi-angular:* Therefore the Angle $ABD = BDC$, wherefore BC is parallel to DC;† so likewise BC is parallel to AD, by reason the Angle ADB is equal to DBC.

117 THEOREM 4. IF two of the opposite Sides of a Quadrilateral are equal and parallel, the other two are also equal and parallel. Eucl. 1. Prop. 23.

THEOREM

* Sup. n. 28.

† B. 1. n. 68.

‡ Sup. n. 27.

§ Sup. n. 80.

* Sup. n. 95.

† B. 1. n. 65.

‡ Sup. n. 75.

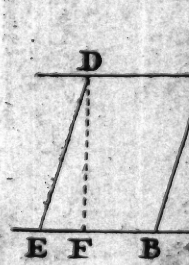
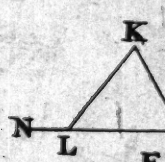
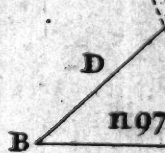
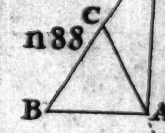
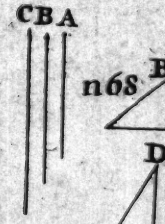
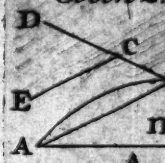
§ Sup. n. 39.

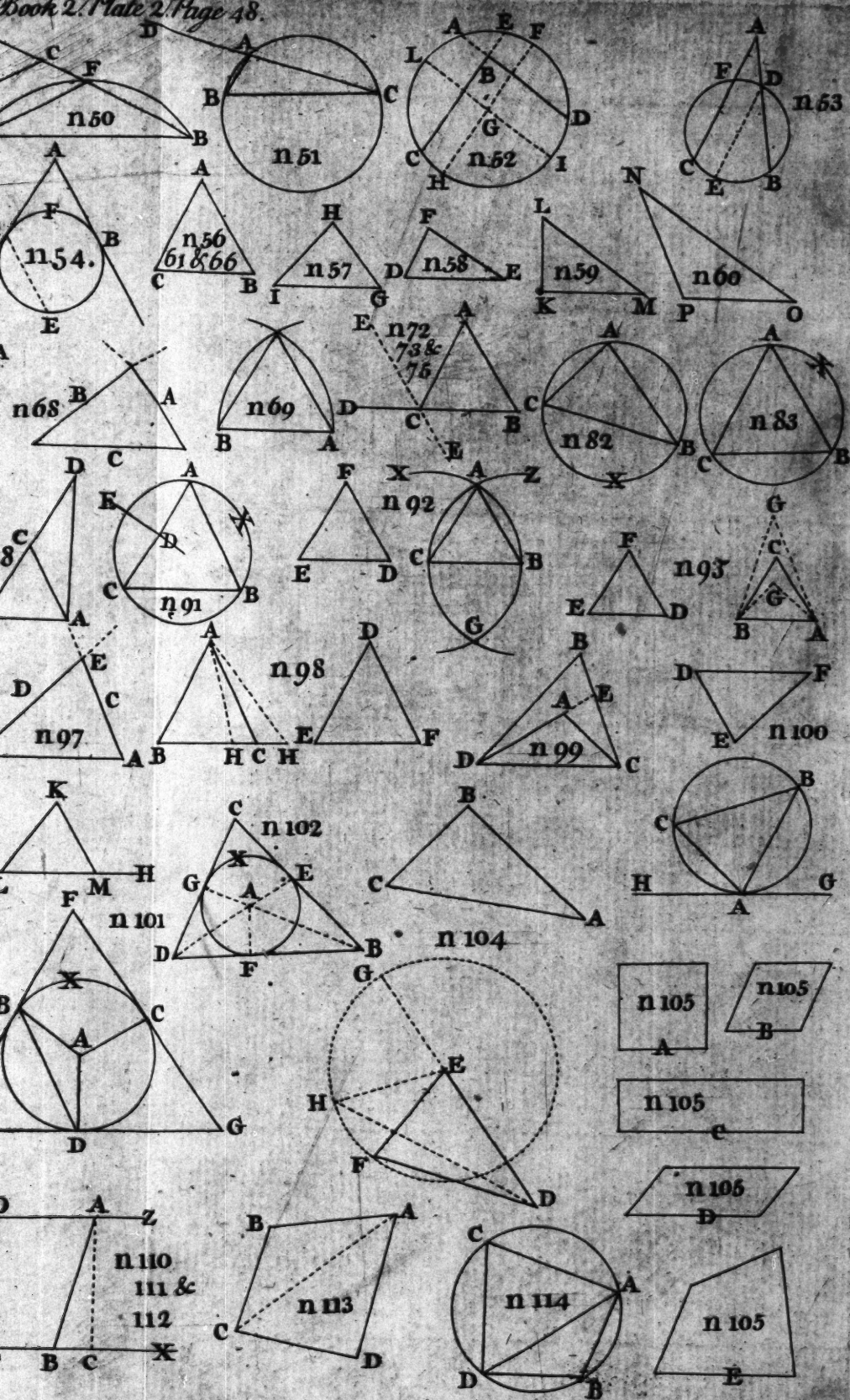
‖ Sup. n. 47.

* Sup. n. 94.

† Sup. n. 26.

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THEOREM 5. If the four Angles of a Quadrilateral are right ones, 118
it is a Parallelogram.

FOR AB and CD by the Hypothesis are Perpendicular upon AC; therefore they are parallel,* AC and BD are also Perpendicular upon DC; consequently by the same reason, they are parallel.

THEOREM 6. THE opposite Angles of a Parallelogram are equal, 119
and those which are contiguous are equal to two right ones.

1. THE Angle $FDA = DCB$,† and $FDA = DAB$,‡ wherefore since two Angles equal to a third, are equal, $DAB = BCD$: Now $FDA + ADC$ makes two right Angles; § then BCD equal to ADF , makes with ADC two right Angles. 2. So likewise it may be demonstrated, that the two opposite Angles ABC and ADC are equal, and that $BAD + ABC$ is equal to two right Angles. Q. E. D.

PROBLEM 1. To make a Parallelogram having one Angle, and 120
the two containing Sides given.

THE given Sides are Z and X, the given Angle is K: Z and X must be joined, so as to make an Angle equal to K,|| and then draw two Lines parallel to Z and to X.¶

COROLLARY. THEREFORE to make a Square having one Side 121
given, there is no more required than to join two Lines equal to that which is given, so as they may make a right Angle. Eucl. 1. Prop. 46.

THEOREM 7. EVERY Polygon, or Figure of several Sides, may 122
be divided into as many Triangles as it has Sides, less two.

IN the Polygonal Figure X, having drawn from one of its Angles as A, right Lines to all the other Angles, it will be thereby divided into several Triangles, whose Bases are the Sides of the Polygon, except two which are at the two Sides of A, of which AB and AF are one of the Sides; hence there is as many Triangles, less two, as there are Sides, 'tis the same in all Polygons; wherefore it may be absolutely said, that a Polygon may be reduced to as many Triangles, as it hath Sides, less two.

COROLLARY. THEREFORE all the Angles of a Figure which hath 123
several Sides, are equal to twice so many right Angles, less four, as the Figure hath Sides.

THAT is, all the Angles of X, which hath six Sides, are equal to eight right Angles; for the Polygon is divided into as many Triangles as it hath Sides, less two, that is, into four. Then since the Angles of each Triangle are equal to two right ones, all the Angles of the four Triangles together, are equal to eight right Angles: But they all together compose the six Angles of the Figure, therefore they are equal to eight right ones; that is, to twelve right Angles, less four, it may therefore be said, that all the Angles of a Chiliagon, that is, a Figure of a thousand Sides, are equal to 1996 right Angles, this may be easily conceived, although it is impossible to conceive a Chiliagon exactly.

H

ADVER.

* B. 1. n. 68.

† Sup. n. 27.

‡ Sup. n. 25.

§ Sup. n. 17. ¶ Sup. n. 29.

|| B. 1. n. 72.

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By regular Figures, we mean those whose Sides and Angles are all equal,* and which may consequently be inscribed in a Circle, the Angle EAF, is called the Angle at the Center, because it is formed at the Center of the Circle by the Radii AE and AF, drawn to the Ends of the Sides of the Polygon, and the Angles similar to GEF, formed by the meeting of the Sides of the Polygon, are called Angles of the Polygon, and as it has been demonstrated,† all the Angles formed at the Point A, which is here taken for the Center of the Polygon, are together equal to four right Angles. When the bigness of each Angle at the Center is required, as here in the Example of the Hexagon, the Sum of four Angles which is 360 Degrees, must be divided by six, the Number of the Sides of the Polygon, the Quotient which is 60, is the content of the Angle at the Center: The Angle at the Center being known, that at the Polygon will be likewise known; for as the three Angles of the Isosceles Triangle EAF, are together equal to two right ones;‡ or 180 Degrees, subtracting therefore the Angle at the Center, the remainder is the Sum of the two equal Angles at the Base, or Side of the Polygon; which are equal to GEF only, the Angle of the said Polygon, hence that of the Hexagon is 120 Degrees: Which might be found also, by what has been demonstrated;§ since that the Angle GEF, is measured by half the Arc of the Circle GHF, upon which it stands, therefore subtracting the Part GF, here supposed 120, from 360 Degrees, which make the whole Circle, there will remain 240, half of which, is the content of GEF, or Angle of the Polygon.

QUESTION.

WHAT Polygons are they, which may be joined by their angular Points, without leaving a Vacuity?

THEY must be such, that the Angles thereof, which are about the common Point, make exactly four right Angles,¶ hence, there is only Triangles, Squares and Hexagons that can do it; the four Angles of four Squares, which have one Point common, make four right Angles, the three Angles of three Hexagons, having one Point common, being each 120 Degrees, they together make 360, the Sum of four right Angles. Six Equilateral Triangles, being placed about a Point, their six Angles touch, and make four right Angles.

WHEN the Angle at the Center of a regular Figure is known, it may be inscribed in a Circle, by drawing from the Center two Radius's, making such an Angle, as the Angle at the Center of the Figure ought to be. For if it is a Figure of ten Sides, which makes an Angle of thirty-six Degrees, which is the tenth Part of 360, the Chord of that Angle will be one of the Sides of that Figure.

To

* Sup. n. 106.
§ Sup. n. 39.

† Sup. n. 19.
¶ Sup. n. 19.

‡ Sup. n. 75.

To Circumscribe a Polygon, or regular Figure about a Circle, it must be first inscribed therein and the Radii be produced, then having divided one of the inscrib'd Sides of the Polygon into two equal parts, as EF, by drawing the Radius AB thro' the middle thereof, and having drawn at the Point B, a tangent between AC and AD it will be one of the Sides of the circumscribing Polygon, then all the Radii of the inscribed Polygon, must be prolonged equal to AC and AD, thro' the Ends of which, having drawn right Lines, you will have the Figure required; which is evident, but the Angle at the Center of every regular Figure, cannot be described with the Rule and Compass only, as shall be made appear, but only mechanically, by means of a Semi-circle divided into Degrees called a *Protractor*.

PROBLEM. 2. To inscribe a Square in the Circle X. Eucl. 4. 124 Prop. 6.

THE Circle given is X, to inscribe therein a Square, thro' the Center must be drawn a Line such as AC, which is a Diameter, thro' the middle thereof draw the Perpendicular BD, the four Points A. B. C. D. are equally distant, * then having drawn right Lines thro' these four Points, there will be a Square in the Circle X. 1. This Figure hath four equal Sides. 2. All the Angles of A, B, C, D, are right ones. † For each of them stands upon the Semi-circumference.

PROBLEM 3. To inscribe a Circle in a Square. Eucl. 4. 125 Prop. 8.

THE Square FGHI is given to inscribe a Circle therein, having divided each of the four Sides of the Square into two equal Parts, and drawn AC and BD, if a Circle be described from the Point E where they intersect, with the extent of AE, it will be inscribed in the Square. For the four Lines AE, BE, CE, DE are equal, hence, the Circle will pass thro' the Points A, B, C, D.

PROBLEM 4. To circumscribe a Square about a Circle. Eucl. 4. 126 Prop. 7.

THE Circle ABCD (*fig. preced.*) is given to be circumscribed by a Square, there must be drawn two Diameters crossing at right Angles, then having drawn thro' the four Ends of these Diameters, four Tangents to the Circle, † they will (*as is evident*) form the Square required.

PROBLEM. 5. To describe a Circle about a Square. Eucl. 4. 127 Prop. 9.

THE Square ABCD is given to be circumscribed by a Circle. Having drawn diagonals, *that is*, right Lines from one Angle to the other opposite thereto, and taken half one of the Diagonals for Radius, A Circle described therewith will be that which is required.

* b. 1. n. 42.

† Sup. n. 44.

‡ b. 1. n. 112.

S E C T. V.

Of the Area of Surfaces.

THEOREM. I.

128 **I**N every Parallelogram, the opposite Sides and Angles are equal one to another, and the Diagonal divides it into two equal parts Eucl. 1. Prop. 34.

LET X be a Parallelogram whose Diagonal is AB: It must be proved, that the Angle ADB is equal to ACB, and that the Sides AC and BD are equal: The two opposite alternate Angles ABD and BAC are equal, * by the same reason $BAD = ABC$, the Side AB is common; hence the two Triangles ABD, BAC are entirely equal; † then the Angle D is equal to the Angle C, and the Angle A to the Angle B, seeing they are composed of equal Angles, and $AD = CB$, as also $AC = BD$, and AB equally divides the Parallelogram ABCD.

129 **THEOREM 2.** PARALLELOGRAMS which are between the same Parallels, and upon the same or equal Bases, are equal. Eucl. 1. Prop. 35 and 36.

If the Parallelograms ABCD, EFCB, are between the same Parallels Z and X, and upon the same Base BC, or another equal thereto, I say, that they are equal. If the Base of the Parallelogram BCEF, is different from that of ABCD, upon BC let there be made the Parallelogram EFCB, † similar and equal to the given one EFCB, it is to be proved, that the last Parallelogram EFCB, is equal to the Parallelogram ABCD. The Sides AB, CD of the Parallelogram ABCD, are Parallel by supposition, they are also equal, § so likewise are the Sides EB, FC. But if ED is added to AD and EF, which are supposed equal, the wholes AE, DF are equal, hence the two Triangles EAB, FDC having their three Sides equal, they are entirely equal; || and subtracting from these two equal Triangles, their common part DGE, the Trapezia's ABGD and CGEF are equal; then adding to each, the same Quantity BGC, which makes the Parallelograms ABCD and BCEF, the two Figures are equal, which was to be proved. Observe that if the Point E, be between D and A, instead of adding DE it must be subtracted, in order to conclude the equality of the Lines AE and DF, and afterwards that of the two Triangles, EAB, FDC, and consequently that of the two Parallelograms, by means of the common Trapezia, which is to be added.

CORAL.

* Sup. n. 25.

† Sup. n. 96.

‡ Sup. n. 120.

§ Sup. n. 128.

|| Sup. n. 92.

COROLLARY. 1. THEREFORE in measuring the Superficies of ¹³⁰
a Parallelogram, regard is had to only the Base, and the Perpendicular Height.

FOR (fig. preced.) the Parallelogram BCEF, is equal to a rectangled Parallelogram whose Base is BC, and whose Sides being Perpendicular, are equal to its Height; thus the rectangled Parallelogram ABCD, is the Measure of all the Parallelograms whose Bases are equal to BC, and which have the same height, or are between the same Parallels X and Z; there can be but one right angled Parallelogram between X and Z upon the Base BC, and there may be an infinite number of Oblique angled Parallelograms upon the same Base, and between the same Parallels X and Z.

COROLLARY. 2. THEREFORE in measuring the extent of an ¹³¹
Oblique angled Parallelogram, no regard is had to its Circumference.

FOR altho' the Sides BE and CF were of a Million, or an infinite number of Miles, which may be conceived supposing the Lines X and Z to be prolonged *ad inf.* the Parallelogram whose Circumference is infinite, is no greater than ABCD whose Circumference is finite.

THEOR. 3. IF two Lines be drawn Parallel to the Sides of ¹³²
a Parallelogram, and thro' any Point whatsoever of its Diagonal; they will divide the Parallelograms into four Parts, of which those two, which the Diagonal does not pass thro', and which are called complements, are equal one to another. Eucl. 1. Prop. 43.

ABC=ADC, and AGF=AKF, and FEC=FHC * Subtracting from the equal Triangles ADC and ABC, the equal Quantities AKF and FHC on one Side, and AGF and FEC on the other, the remainders FKDH and BEFG, are equal.
Q. E. D.

THEOR. 4. PARALLELOGRAMS are double the Triangles of the ¹³³
same Base and Height. Eucl. 1. Prop. 41.

THE Triangle ECD hath the same Base as the Parallelogram ABCD, and they being both between the Parallels X and Z, have the same Altitude; draw DF Parallel to CE, making the Parallelogram DCEF, which is equal to ABCD. † Now CED is equal to EDF, ‡ therefore DCEF or its equal ABCD, is the double of CED.

COROLLARY. 1. THEREFORE Triangles of the same or equal ¹³⁴
Bases, and having the same height, are equal. Eucl. 1. Prop. 37 and 38.

By reason they are each the half of a Parallelogram of the same Base and height.

COROLLARY. 2. THEREFORE in measuring a Triangle, regard ¹³⁵
is had to only its Base and height.

* Sup. n. 128. † Sup. n. 129. ‡ Sup. n. 128.

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To measure the Triangle ABC, there must from the Vertex be let fall a Perpendicular AD, which is the Altitude thereof, and BC must be multiplied by half AD, or AD by half BC, or BC by AD, and that Product divided by 2, all which amount to the same. If the Triangle were obtuse-angled, as MNP, then in order to obtain the height thereof, produce the Side MP, upon which let fall a Perpendicular NQ, from the Vertex N.

BUT if the three Sides of the Triangle are known, its Content may be found without drawing a Perpendicular, thus. Add the three Sides together into one Sum, and from half that Sum, Subtract each Side severally, so will there be three Remainders, then multiply the half Sum of the three Sides, by one of those Remainders, and that product by another, and this last product by the third or last Remainder, then will the Square-Root of this last Product, be the Area, or Content of the Triangle required,

- 136 THEOR. 6. EQUAL Triangles having the same or equal Bases, and placed on the same Side, are between the same Parallels, Eucl. I. Prop. 39 and 40.

If they are equal, they have the same Altitude, * and Consequently the Perpendiculars let fall from the Vertex of each, to its corresponding Base, are equal. Therefore if a Line be drawn thro' the Vertices of these two Triangles, it will be parallel to the Bases thereof. †

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- 137 If there are two Triangles whose Superficies are Equal, that which hath the most Sides and Angles equal one to another, hath the least Circumference, a right angled Triangle, hath a greater Circumference than an Equilateral one which is equal thereto; for let ABC be a right angled Triangle, and ABD an Equilateral one having the same height, they are Equal, † Prolong AC to E, making $AC=CE$, the Circumference ABC is $AB+BC+CE$. Produce BD to E, each Angle of ABD is 60 Degrees, and EAB being Ninety, the Angles AEB and EAD are each 30, therefore EDA is an Isosceles, and $DE=AD$, then $AB+BE$ is the Circumference of ABD; now BE is shorter than $BC+CE$; therefore the Circumference of ABD is shorter than that of ABC, The more regular and uniform a figure is in all its parts, the greater is its contained Surface, and the lesser its Circumference. As will appear farther on.

- 138 PROBLEM. 1. To describe a Parallelogram having one Angle thereof given, equal to a Triangle given. Eucl. I. Prop. 42. The

* Sup. n. 134.

† b. I. n. 64.

‡ Sup. n. 134.

THE given Triangle is ABC, and the given Angle D; required to make a Parallelogram having an Angle equal to D, and equal to the Triangle ABC. 1. Thro' B the Vertex of the Triangle, draw BF parallel to its Base AC. * 2. Upon E the middle of that Base, draw GE making with AC an Angle equal to D. † 3. Compleat the Parallelogram CEGF, ‡ which will be that which was required; for it is equal to the Triangle ABC, § and hath the Angle GEC equal to the given Angle D.

PROBLEM 2. To describe upon a given right line, a Parallelogram equal to a given Triangle, having one Angle equal to a rectilinear Angle given. Eucl. 1. Prop. 44. 139

THE given Line is A, the given Angle is C, required to make a Parallelogram equal to the Triangle B. 1. By the preceding Problem, make the Parallelogram DEFG equal to the Triangle B, having an Angle equal to C. 2. Prolong GF and DE, so that FH = A, and EI = FH; compleat the Parallelogram EFHI, and produce the Line IF till it meet the produced Part of DG, and compleat the Parallelogram DKLI, in which is FHLM equal to DEFG; || hence FHLM is the Parallelogram required, equal to the Triangle B (to which DEFG was made equal,) having an Angle equal to the given Angle C, and being made upon FH equal to the given Line A.

PROBLEM 3. To describe a Parallelogram equal to a given right lined Figure, and having one Angle equal to a given right lined Angle. Eucl. 1. Prop. 45. 140

THE given right lined Figure is ABCD, the given Angle E. 1. Reduce the Figure ABCD into two Triangles, by the Line AC. 2. Draw the Parallelogram GI equal to the Triangle ABC, having the Angle G equal to the given Angle E. * 3. Upon HI draw also the Parallelogram IL equal to the Triangle ACD, having the Angle IHL equal to the said Angle E, † this being done, GK = GI + IL; therefore GK = ABCD is the Parallelogram required, But if the Figure had been composed of more than two Triangles, the same thing would have been done upon the Line KL for the third Triangle, as was done upon HI for the Second; and so likewise for the rest.

DEFINITION. THE Side opposite to the right Angle of a rectilinear Triangle, is called Hypotenuse. 141

THEOREM 6. IN every rectangled Triangle the Square of the Hypotenuse, or side which subtends the right Angle, is equal to the Squares of both the other Sides. Eucl. 1. Prop. 47. 142

LET the rectangled Triangle be CAB, if upon the Hypotenuse CB be made the Square CD, (a Parallelogram or Square is denoted by two diagonal Letters) and upon the other two Sides CA and AB, be made the Squares CK, BG: I say, that $CD = CK + BG$. From the Vertex A, let there be drawn the Line AL

Pet.

* b. 1. n. 72. † Sup. n. 29. ‡ b. 1. n. 72.

§ Sup. n. 133. || Sup. n. 132.

** Sup. n. 138. † Sup. n. 139.

Perpendicular upon the Hypotenuse BC, which will divide the Square CD into the two Parallelograms IE and ID, the first of which is equal to the Square CK, and the second to the Square BG; which is thus demonstrated. From the Point A to the Point E; let there be drawn the Line AE, and from B to the Point H the Line BH, these two Lines will form the two equal Triangles ACE, ECH,* for the Sides $CE = CB$, and $CA = HC$, and the Angles ACE and HCB which those Sides contain are equal, having each a right Angle, and the Angle ACB common; but the Triangle ACE is half the Parallelogram IE,† because they have both the same Base and are both between the same Parallels, the Triangle HCB is likewise for the same Reason, half the Square CK, now these Triangles are equal; therefore the Parallelograms of which they are the Halves, are also equal.

WHAT has been said of the Parallelogram IE, and of the Square CK, is to be understood of the Parallelogram ID, and of the Square BG; therefore, &c.

PROBLEM 4. To describe a Square equal to two, or to several other Squares.

JOIN the two Bases AB, and AC, of the two Squares proposed, so that they make a right Angle, the Base CB of that Angle will be the Side of a Square equal to both the Squares, according to the preceding Theorem. If a Square be required equal to three given Squares, and that BD is the Side of the third, join BC and BD, so that they make a right Angle; then the Square of CD is equal to the Squares of BC and BD, consequently to the Squares of AC, AB, BD; and by this means a Square may be formed equal to as many Squares you please.

THEOREM 7. If the Square of one of the Sides of a Triangle is equal to the Squares of the other two Sides, the Angle contained between those two Sides is a right one. Eucl. 1. Prop. 48.

I suppose the Square of the Side BD of the Triangle ABD, to be equal to the Squares of the other two Sides AB, AD, if so, I say, that the Angle BAD contained between the two Sides AB, AD, is a right one. Draw AC Perpendicular upon AD and make it equal to AB, then the Squares of AD and AC are equal to that of DC,* now by the Hypothesis, these two Squares are equal to that of BD; the Squares of BD and DC are therefore equal. Hence CD and BD are equal; then the two Triangles ABD and ADC are entirely equal, and consequently Equi-angular,† BAD is therefore a right Angle as well as CAD, it is here supposed that equal Squares have equal Sides, which is evident.

THEOREM 8. A Triangle is equal to two or more Triangles having the same Height, if its Base be equal to all their Bases taken together.

THE Triangle ABC is equal to the Triangles ABE, EAD and DAC, which are its Parts. Now $ACD = CFD$, and $DAE = DGE$, and $EAB = EHB$,‡ therefore CAB is equal to two or more Triangles, &c. which was to be proved.

DEFI-

* Sup. n. 98.

† Sup. n. 133.

* Sup. n. 142.

‡ Sup. n. 92.

‡ Sup. n. 134.

DEFINITION. THE Part of Radius which is Perpendicular between the Side of a Polygon, and the Center of the Circle, is called Apotome. 146

THEOREM 9. THE superficial Content of a regular Polygon, is equal to a Triangle whose Base is equal to the Circumference of the Polygon, and its Height equal to the Apotome thereof. 147

HAVING drawn Lines from the Center A to each Angle, it will be reduced into as many Triangles as it hath Sides, all equal to the Triangle ABC whose Base is BC, and its Height the Apotome AD, now a Triangle whose Height is AD, and its Base the Circumference of the Polygon, is equal to all these Triangles whose Height is DA; and the Bases taken together, make the Circumference of the Polygon; * which was to be proved.

EVIDENT Propositions concerning Polygons.

PROP. 1. A Polygon is greater than its inscribed Circle. 148

PROP. 2. A Polygon is lesser than its circumscribing Circle. 149

PROP. 3. A Circle may be esteemed as a Polygon having an infinite Number of Sides. 150

FOR if a Polygon of a small Diameter hath a Million of Sides, it is evident, that the difference between it and a Circle is not preceivable, if I say, there be imagined in a Circle as many Sides as there are visible Points, it will be a Polygon and a Circle at the same Time.

THEOREM 10. OF two regular Polygons which circumscribe a Circle, that which hath most Sides, hath the least Circumference and the least Surface. 151

X is a Polygon which circumscribes a Circle, I divide its Sides in order to make another Polygon having more Sides, by drawing Tangents which are without the Circle † hence the Polygon will be greater than the Circle, ‡ let us consider the same Part of the two Polygons, that is, the Part which circumscribes the same Part of the Circle, for Example BFD, since BA + AC is longer than BC, then BD + BA + AC + CF is longer than DB + BC + CF: Hence the Circumference of that which hath fewest Sides is greatest. The Figure EFCABD exceeds EFCBD by the bigness of the Triangle ABC; therefore the Surface of that which hath most Sides is least.

COROLLARY. THEREFORE since the more Sides a circumscribing Polygon hath, the lesser it is, (always remaining greater than the Circle which it circumscribes) it follows, that the more Sides a Polygon has, the nearer its Circumference and Surface approach to the Circumference and Surface of the Circle which it circumscribes, and that therefore a circumscribing Polygon having an infinite Number of Sides, is the same as a Circle. 152

THEOREM 11. IF there are two regular Polygons inscribed in the same Circle, that which hath the most Sides, hath the greatest Circumference, and the greatest Surface. 153

I

Two

* Sup. n. 143.

† b. I. n. 106.

‡ Sup. n. 146

Two Polygons being inscribed in the Circle X; let us consider the part ABDC, and the corresponding Parts of the two inscribed Polygons. 1. $BD + DC$ is longer than BC , wherefore the Circumference of the Polygon having most Sides, appears already to be the longest. 2. The Figure $ABDC$ exceeds ABC , by the bigness of the Triangle BDC , therefore the Polygon which hath most Sides hath the greatest Surface, which was to be proved.

154 COROLLARY. THEREFORE since two or more Polygons inscribed in the same Circle, that which hath most Sides is greatest, so long as it remains lesser than the Circle, it from hence follows, that the more Sides an inscribed Polygon hath, the nearer it approaches to the Circumference and Surface of the Circle, and that therefore an inscribed Polygon, the Number of whose Sides is infinite, is the same as the Circle.

155 THEOREM 12. OF two regular Polygons inscribed in one and the same or equal Circles, that which hath most Sides hath the longest Apotome.

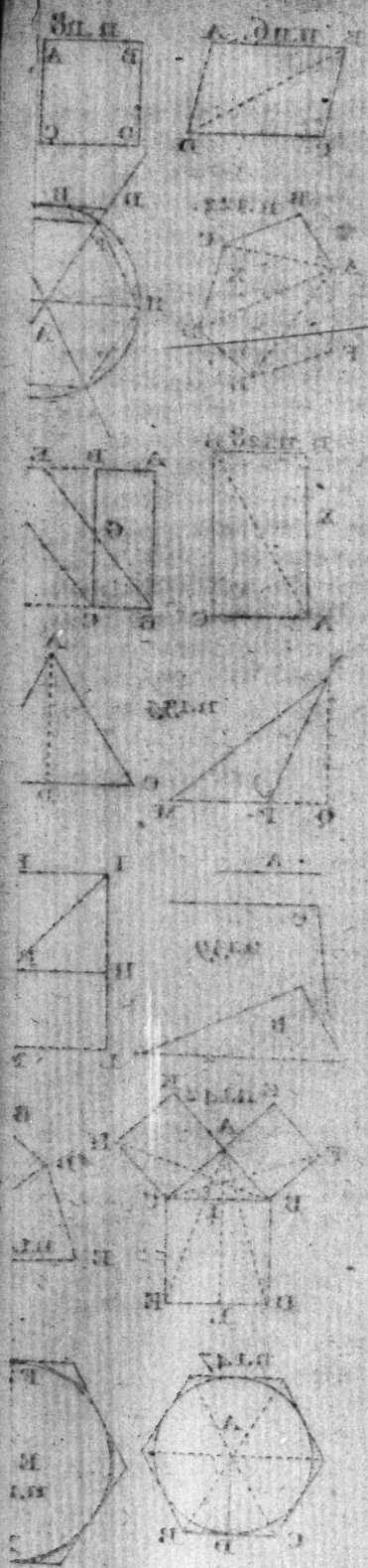
EC is the Chord of the Polygon which hath most Sides, CL the Chord of that which hath the fewest, hence CE being the Chord of a lesser Arc, is therefore shorter, and consequently farther from the Center A than CL ,† then the Perpendicular AD which is the Apotome of the Polygon whose Chord is CE , is longer than the Apotome of the Polygon whose Chord is CL , which was to be proved.

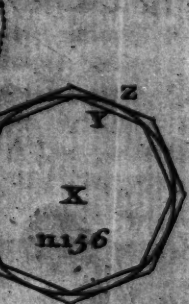
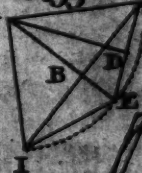
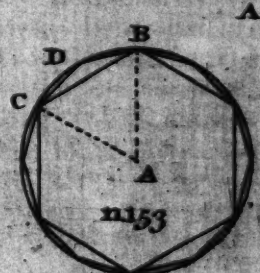
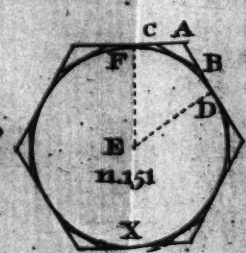
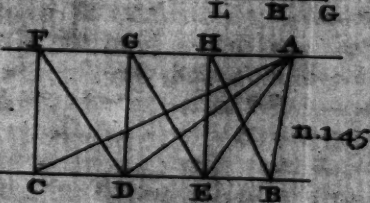
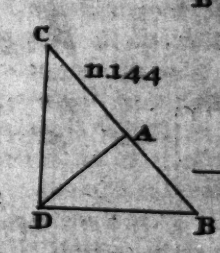
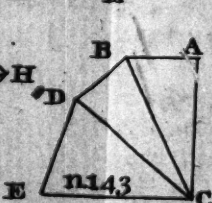
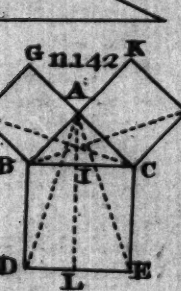
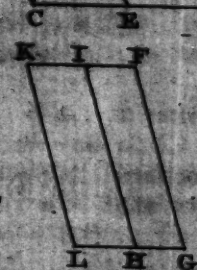
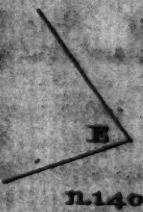
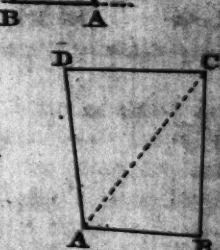
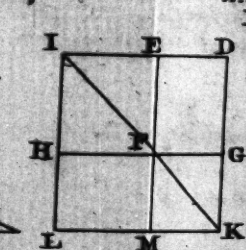
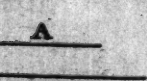
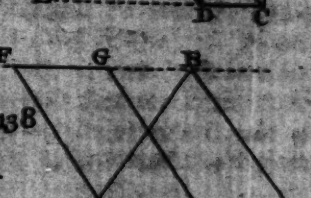
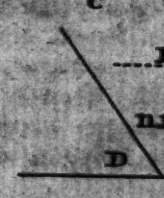
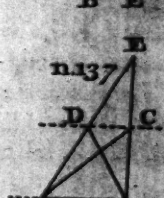
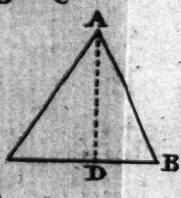
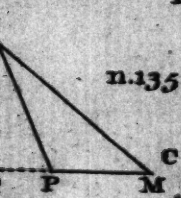
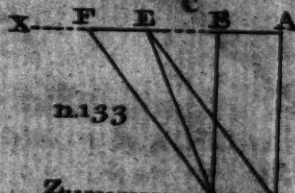
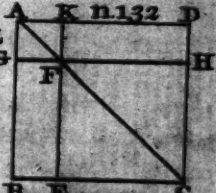
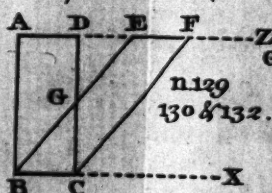
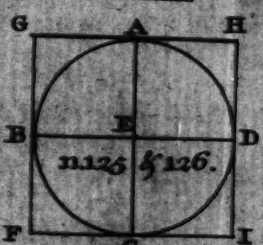
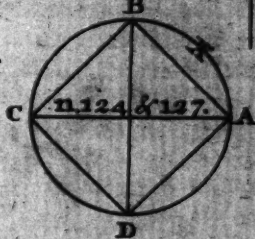
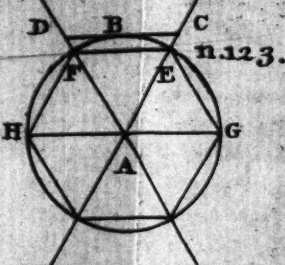
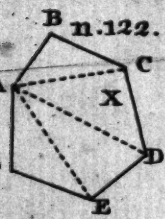
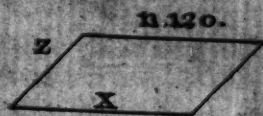
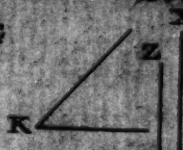
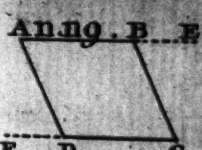
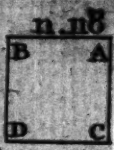
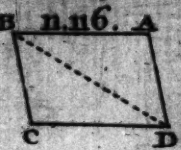
156 THEOREM 13. THE Surface of a Circle is equal to a Triangle whose Height is the Radius, and its Base the Circumference of the Circle.

ACCORDING to the two preceeding Theorems and their Corollaries, a Circle may be supposed as equal to a circumscribing or inscribed Polygon, the Sides whereof are infinite. The Surface of such a Polygon, is equal to a Triangle whose Base is the Circumference thereof, (which is the same thing as the Circumference of the Circle,) and whose Height is the Apotome, or Perpendicular drawn from the Center of the Polygon upon one of the Sides thereof, the Sides of which being infinite, that Perpendicular will be the same as the Radius of the inscribed Circle; taking therefore the Circle for a similar Polygon, its Surface is equal to a Triangle, whose Base is equal to its Circumference, and the Height equal to the Radius thereof.

THE Radius of a Circle may be easily measured, but how to measure the Circumference remains yet unknown, which prevents our finding the Quadrature of the Circle, that is, of making a Square equal to the Surface contained in a Circle, for if the Circumference were known, it would be easy to form a Triangle equal to the Circle, by making its Base equal to the Circumference, and its Perpendicular equal to the Radius, the Triangle (as will hereafter appear) may be easily transformed into a Square equal to it, and consequently equal to the said Circle, but the Circumference of the Circle can be no other-

ways



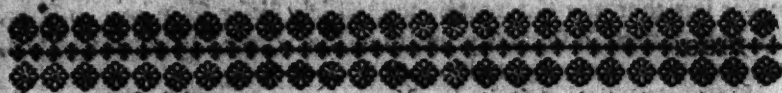


ways expressed, than by assigning two Lines one greater, and the other less than the Circumference, which differ from one another, by only a certain Quantity less than the whole Quantity signified, which is thus demonstrated.

LET the given Circle be X, and the supposed Polygon Z, which circumscribes the Circle, and another which I call Y is inscribed therein, let the given Quantity, which is the Difference between the Surface of the Circle to any other Surface, be T. Increase or diminish the Number of the Sides of the Polygons Z and Y, till their Difference is less than the Quantity T, which is easily done, for by augmenting the Sides of each, the Bigness of Y is encreased,* and that of Z diminished;† hence each of them approach nearer to the Circumference of the Circle; the Difference between Z and Y is greater than that of Z to the Circle X, since X is greater than Y, then the Difference between the Surfaces Z and Y, being lesser than the Quantity T, there is found a Surface which differs from that of the Circle, by a Quantity lesser than what was proposed, that is: if it were proposed to find a Surface differing from that of a given Circle, by only the hundred thousandth Part of a Line, there may be found another that differs yet less.

* *Sup. n. 152.* † *Sup. n. 149 and 150.*





B O O K. III.

THE Properties belonging to Quantity in general,
applied to Lines, Planes and Solids.

S E C T. I.

THE four Operations of Arithmetic, ADDITION, SUBTRACTION, MULTIPLICATION, and DIVISION, upon Lines, Planes and Solids.

A D D I T I O N.

THE first and simplest Property of Quantity in general, and consequently of a Line, a Plane, and a Solid, is that it may be either augmented or diminished, one Line may be added to another, by joining or prolonging them, so likewise a Plane to a Plane, by continuing or putting their Sides together, when Quantities are expressed by Letters, whether they be Lines, Planes, or Solids, the Sign of Addition is +, thus $a+b$ signifies that the Quantities which are called a and b , (*whether they be Numbers or Lines*) are added together. When the same Letter is repeated, or taken several Times, it is written only once, but there is a Figure put before it, which shews how often it is added to itself; $3\ b$ signifies that b is added three Times to itself. Those Quantities which are the same, and have different Signs are erased, or cancelled. In Geometry a Line is generally denoted by two Capitals placed at the Extremities thereof, as AB , or by only a small Letter which is put at the Middle:; thus a denotes the Line AB .

A ————— B

a

S U B T R A C T I O N.

THE Sign of Subtraction is —, thus $a-b$ signifies that b is to be subtracted from a , and that consequently a is greater than b . The Difference of two Quantities is the Excess of the greater above the lesser, or the greater, less the lesser.

Let there be two Lines AB and CD , having A ——— E ——— B
taken from AB , the Line AE , equal to CD , C ——— D
their Difference is EB , which is the Excess of AB above CD , or
the greater Line AB , less the lesser Line CD , that is, what re-
mains

main

mains of the greater, after the lesser is deducted. If the Plane BCEF, be subtracted from the Plane ABCD, the Subtraction is denoted thus, $ABCD - BCEF$, which is the Difference of the two Planes, after the Subtraction the remainder is the Plane ADEF, the general Rule of Subtraction is this; *change all the Signs of the Quantities that are to be subtracted: that is, change Plus into Minus, and Minus into Plus, the Sign + is always understood as before the Quantity which hath no Sign, thus to subtract b from a, that is, + b from a, the Sign + must be changed into —, and it must be wrote $a - b$.*

MULTIPLICATION.

To multiply one Quantity by another, is to take one, (*it matters not which first*) as many Times as there are Unites in the other. To multiply a by b, is to take a as many Times as b hath Unites or Parts, which is expressed by joining the two Letters thus ab. When a Plane is marked by two Letters, as ab: it is supposed that a, or BC, makes a right Angle with b, or AB: In the preceeding Book we demonstrated, that the bigness of a Plane does not depend upon the Length of its Sides only, but also upon the Nature of the Angle which is formed thereby, that a right angled Parallelogram is greater than an Oblique one: * Hence there is a determinate Measure, for which Reason it is supposed, (*as I shall proceed to shew*) that two Lines which are conceived as multiplied one by the other, make a Rectangle. When small Letters are used, if they are joined together, it denotes that they are multiplied into each other, but it is not so when Capitals are used; thus AB does not denote that A is multiplied by B, but that AB is a Line whose Extremities are A and B, the Sign of Multiplication of two Lines marked with Capitals is this X, that is, $AB \times BC$, signifies that AB is multiplied by BC. If it be conceived that the Plane ABCD be taken and put upon itself as many Times as there are Parts in DE, it will make a Solid: $ABCD \times DE$, if $AB = a$, and $AD = b$, and $DE = c$, the Product will be abc, as a Plane is denoted by two Letters, so a Solid is denoted by three.

DIVISION.

THE Quantity that is proposed to be divided, is called *Dividend*: That which is divided by, is called *Divisor*; and that which shews how many Times the Divisor is contained in the Dividend, is called *Quotient*.

To shew that one Quantity is divided by another, the Letters by which they are represented, are put under one another, and separated by a little Line, the Dividing a by b, is denoted by putting the a above b in this Manner $\frac{a}{b}$, which shews that a is to be conceived as divided by b. When there are Letters equally contained in the Divisor, and Dividend, they must be cancelled, thus to divide cd by c, the c must be suppressed and d remain, which

* B. 2. n. 230.

which will be Quotient of cd divided by c , that is, d signifies how many Times c is contained in cd .

- 5 THE same Operations repeated upon compound Quantities. A Quantity is called compound, when it is composed of two, or more Quantities, which are each expressed by a particular Sign, $a+b$, and $a-b$, are compound Quantities; more one Quantity less the same Quantity is nothing, $+3-3$, is equal to Cypher, therefore to render an Expression elegant, those Quantities which are the same, but have contrary Signs, are cancelled: Having for Example this Expression $5b-2b$, it is reduced to this $3b$, for as was before said, when a Quantity has no Sign expressed, the Sign $+$ must be understood, thus $5b-2b$, is the same as if it was $+5b-2b$; now in this Expression there is $2b$ with a contrary Sign, therefore I cancel it, and write $+3b$, or simply $3b$.

ADDITION of compound Quantities.

- 6 Compound Quantities are added in the same Manner as simple ones; to add $b+c$ to $f+h$, they must be joined by the Sign $+$, thus $b+c+f+h$, Letters which are the same, but have contrary Signs, must always be cancelled, thus in adding $b+d$ to $c-d$, the Sum is only $b+c$. When the same Letter is repeated several Times, there is put only one, with a Figure before it, shewing how many Times it is taken, as here to add $4f+6g$ to $3f-4g$, is wrote $7f+2g$, if this Operation had been expressed at length, there would have been wrote, $4f+6g+3f-4g$; now $4f+3f=7f$, and $+6g-4g=+2g$. A Quantity having the Sign $-$, being on one side of the Sign $=$, if you would add it to the other side, cancel it from the Place where it stands, and put it on the other side with the Sign $+$, thus let $a=b-d$, to add d to the other side, make it $a+d=b$, this expressed at length is $a+d=b-d+d$, but $-d+d=0$.

SUBTRACTION of compound Quantities.

- 7 THE general Rule for this is the same as for simple Quantities, the Signs of the Quantity required to be subtracted must be changed, the Sign $+$ ought always to be understood as before every Quantity which has no Sign, therefore to take $b+d$, or $+b+d$ from $c+f$, there must be wrote $c+f-b-d$. For 1. b must be joined to $c+f$ with the Sign of Subtraction. 2. 'Tis not only $+b$ that should be subtracted, but also $+d$, it must therefore be wrote $-d$, on the contrary to take $b-d$ from $c+f$, the Signs of $+b-d$, must be changed, putting them $c+f-b+d$.

OBSERVE well in this Expression $c+f-b-d$, it is not from b that d is subtracted, but b and d are subtracted from $c+f$. When one and the same Quantity is to be subtracted from each side of the Sign of Equality, it is only cancelling it in the Place where it is with the Sign $+$, and to put it on the other side with the Sign $-$, let $a=b+d$, to take away d from each side, write $a-d=b$.

MULTI-

MULTIPLICATION of compound Quantities.

THE Questions in Multiplication of compound Quantities, are reduced to these three. 1. To multiply a Line *Plus* another Line, by a Line *more* another Line, as $a+b$ by $f+g$. 2. To multiply a Line *Minus* another Line, by a Line *Plus* another Line, as $a-b$ by $f+g$. 3. To multiply a Line *Minus* another Line, by a Line *less* another Line.

RULES for these Three Cases.

RULE 1. WHEN the two Quantities given to be multiplied together, have each the Sign $+$, their Product ought to have the same Sign $+$.

To multiply $a+b$ by $f+g$, the Operation must begin by multiplying a by f , writing af , which signifies the Product of a by f , making in like Manner as many particular Multiplications as there are Letters, it will be $af+bf+ag+bg$, which is the Product of $a+b$ by $f+g$. Let $a+b=AC$. Let $a=AB$, and $b=BC$, also let $f+g=AG$, and $f=AH$, and $g=HG$, I suppose that $ACEG$ is a Rectangle, cut by Parallels which make the Parallelograms $ABIH$, $FGIH$, $BCDI$, $DEFI$, which several Parallelograms are together equal to their Sum $ACEG$, the Whole being equal to its Parts; now these four Products $af+bf+ag+bg$, are equal to these four Parallelograms, as is evident, they are therefore equal to $AC \times AG$, that is, to the Parallelogram of AC by AG , or of $a+b$ by $f+g$.

RULE 2. MORE into less, or less into more, gives a Product which ought to have the Sign $-$.

THAT is, if one of the two Quantities hath the Sign $-$, and the other the Sign $+$, their Product ought to have the Sign $-$, to multiply according to this Rule, $a+b$ by $f-g$. 1. Multiply $a+b$ by f , which makes $af+bf$, but as $a+b$ should not be multiplied by the whole Value of f by the Product $af+bf$ is too much the Value of g , that is, by $ag+bg$ I therefore take away what I had put too much, and the Expression will be $af+bf-ag-bg$; let $a=AB$, or HI , $b=BC$, or ID , $g=HG$ or IF , and $f=AG$, hence $a+b=AC$, and $f-g=AH$; then $af=AB \times AG$, and $bf=BC \times BF$, it must be demonstrated, that $af+bf-ag-bg=ACDH$, or that $ACDH=ACEG-FGHI-DEFI$, which is evident.

RULE 3. LESS into less gives the Sign $+$.

IN multiplying these two compound Quantities together, $a-b$ and $f-g$, their Product is $af-bf-ag+gb$, in which you see that the Product of $-g$ by $-b$, is marked with the Sign $+$, to prove that this is right. Let $a=AC$, and $b=BC$, hence $a-b=AB$, let also $f=AG$, and $g=HG$, hence $f-g=AH$, to demonstrate what is required; consider that as $ABIH$ is equal to $ACEG$, or af , if there was subtracted, 1. $DEGH$, or ag , and $BCEF$, or bf , provided that $DEFI$, or bg , which was taken away too much be restored thereto, for in deducting $DEGH$ and $BCEF$,

BCEF, twice DEFI, or bg , is deducted, hence $af - bf - ag + bg$ is the Product of $a - b$ by $f - g$; therefore less into less gives $+$, that is, at the End of the Product there ought to be the Sign $+$, because that having taken away too much, the overplus must be returned.

DIVISION of compound Quantities.

- 9 THE general Rule for Division, whether the Quantities be simple or compound, is to put the Dividend above a Line and the Divisor under the same, thus to divide $ax + cd$ by $xd + cb$, I write $\frac{ax + cd}{xd + cb}$, it was shewn in simple Division, that the same Letters being found both above and below must be cancelled, the same must be also done here, when they are found in each Part of the Dividend and of the Divisor. This Expression $\frac{ax + dx}{bx + cx}$ may therefore be reduced to this $\frac{a + d}{b + c}$, which is simpler and of an equal Value.

As Division undoes what Multiplication does, the three Rules that were given in Multiplication, suffice to shew what Signs the Quotient ought to have, 1. Whether the Dividend and Divisor have both the Sign $+$, or 2. Whether one be $+$, and the other $-$, or 3. Whether both have the Sign $-$.

S E C T. II.

OF the Power of Lines.

DEFINITION I.

- 10 THE first Power of a Line, is the Line itself before it is multiplied.

- 11 DEF. 2. THE second Power, or Square of a Line, is the Product of the Line multiplied by itself.

THE second Power of b is bb , or b^2 ; the Number 2 denotes the two Dimensions of bb . Take Notice, that there is a great Difference between b^2 and $2b$; for $2b$, shews that b has been added to itself, whereas b^2 , signifies that it is a Square, or that b is multiplied by itself; which is very different, for Example 3 and 3, or 3 added to itself makes 9. When a Line is marked at its Ends with two Capital Letters as A and B , its Product or second Power, which is the Square $ABCD$, is marked with four Letters, or else by two, which are at the Ends of the Diagonal as A and C , or the same Line AB joined to itself with the Sign of Multiplication thus $AB \times AB$, or in short, a Line is drawn above AB and the Sign of the second Power joined thereto, in this Manner

$\overline{AB}^2$ $\overline{MN}^2$
 $\overline{AB}^2$, thus MN is a Square whose side is MN , or which is produced by MN multiplied by MN . AD.

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EUCLID and the ancient Geometricians took the Square for the first Power, which is esteemed by the modern Geometers to be the second. A positive Quantity before it be multiplied by itself, or by another, hath one Value, or one Power. Quantities which have the Sign +, are called Positive, and such as have the Sign —, are called Negative.

DEF. 3. A Line which being multiplied makes a Power, is called the Root of that Power.

THE square Root of bb or b^2 , is a Line equal to b , that is, to the side of a Square equal to bb .

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WHEN the Root of a Power cannot be expressed by Number, which happens in certain Cases, as shall be demonstrated, this Sign $\sqrt{\quad}$ is put before it, with the Sign of the Power thus $\sqrt{\quad}^2$, if it is a square Root; if it be a Cube Root thus $\sqrt{\quad}^3$.

DEF. 4. THE Cube, or third Power of a Line, is the Product which it makes, when it multiplies its Square.

THE Square of b is bb , if bb is multiplied by b , which makes bbb , the Solid or Cube is the third Power of the Line b , the Cube bbb may be expressed thus b^3 , the Number 3 denotes its three Dimensions; b^3 is different from $3b$, for $3b$ is b taken or added to itself three Times and b^3 is multiplied once by itself, which produces its Square bb , or b^2 , and secondly, that Square is multiplied by b , which makes bbb , or b^3 , the Number 4 taken three Times, makes 12: But the Square of 4, (which is 16) multiplied by its Root, (which is 4) makes 64. When a Line as AB , is marked at its Ends by two Capital Letters, its Cube or third Power may be expressed thus; $AB \times AB \times AB$, which shews, 1. That AB is to be multiplied by AB , which gives the Square of AB . 2. That the Square of AB is to be multiplied by AB , the Root of that Square, this Expression is abbreviated by putting a Line above AB with a little Figure, shewing the

Number of its Dimensions; thus AB^3 is the Cube of AB .

DEF. 5. EVERY Quantity which is composed of the Product of two other Quantities, is called a Plane.

THUS bd is a Plane, or plane Quantity, one of the Letters whereof denotes the Breadth, and the other the Length, that is, its two Dimensions; $AB \times BC$ is a plane Quantity, or simply a Plane.

DEF. 6. THREE Quantities multiplied together produce a Solid.

THUS bcd is a Solid, whose three Letters denote the three Dimensions thereof, the Product of two of these Letters denote the Plane or Surface, and the third denotes the Height, or Thickness by which the Plane hath been multiplied.

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By what order soever two Quantities are multiplied, the Product is always equal: ab and ba make the same Plane; it is the same of three or more Lines: abc , bca , cab , by what order soever they are multiplied, the Product is the same. In Euclid, the Word Rectangle signifies a right angled Plane; the Planes and Solids hereafter treated of, are all supposed to be Rectangles.

- 17 PROP. 1. IF of two right Lines, one is cut into as many Parts you please; the Rectangles comprehended under the whole Line, and each Part of the divided Line, are equal to the Rectangle made by both the whole Lines. Eucl. 2. Prop. 1.

LET AE and AB be two right Lines: AE is cut into the Parts AC , CD , DE , the Line AB is undivided; it must be proved, that the Rectangle made by the whole Lines AB and AE , is equal to the Rectangles made by the whole Line AB , and each of the Parts of AE ; that is, that $AB \times AE = AB \times AC + AB \times CD + AB \times DE$. The Parts being together equal to their Whole, AE is the same, as $AC + CD + DE$, consequently 'tis the same thing, to multiply AB by AE , as by $AC + CD + DE$; therefore $AB \times AE = AB \times AC + AB \times CD + AB \times DE$; Q. E. D.

- 18 PROP. 2. IF a right Line be any how divided, the Rectangles comprehended under the whole Line, and each of the Parts, are equal to the Square of the whole Line. Eucl. 2. Prop. 2.

LET the Line AB be cut into two Parts at the Point C , let $AB = a$, and $AC = b$, and $BC = d$, the Whole being equal to its Parts, $a = b + d$, therefore to multiply by a , and to multiply a by $b + d$, is doing the same thing; and consequently $aa = ab + ad$; that is, the Square of the whole Line AB or aa , is equal to the Rectangles made by the whole AB , and AC , and AB , and CB ; which was to be proved.

- 19 PROP. 3. IF a Line be any how divided into two Parts, the Rectangle comprehended under the whole Line, and one of its Parts, is equal to the Rectangle of both the Parts, together with the Square of the first Part. Eucl. 2. Prop. 3.

LET the Line AB be any how divided into two Parts at C , it must be demonstrated, that $AB \times AC = AC \times BC + AC^2$.
 Let $AB = a$, and $AC = b$, and $BC = d$, hence $A \overset{b}{\text{---}} C \overset{d}{\text{---}} B$
 as $AC + CB = AB$, so likewise $a = b + d$: then
 multiplying a and $b + d$ by the same Multiplier b , the Products are equal; $ab = bd + bb$; now ab , is the Rectangle of AB by AC , and $bd + bb$ is the Rectangle made of the Parts b and d , together with the Square of the first Part b ; Q. E. D.

PROP.

PROP. 4. IF a right Line be any how divided into two Parts, I 20
say, that the Square of the whole Line, is equal to the Square of
each Part, and to twice the Rectangle contain- A ——— B
ed under both these Parts. Eucl. 2. Prop. 4.

Let the Line AB be divided into two Parts at the Point D, it
must be proved, that $AB^2 = AD^2 + 2AD \times DB + DB^2$. Let $AD = b$,
and $DB = d$, then $AB = b + d$. Now the Square of $b + d$, is $b^2 +$
 $2bd + d^2$, † but $AD^2 = bb$, and $DB^2 = dd$, and $2AD \times DB = 2bd$,
therefore $AB^2 = bb + 2bd + dd$; which was to be proved.

PROP. 5. IF a Line be divided into two equal Parts, and into two 21
unequal ones, the Rectangle contained under the two unequal Parts,
together with the Square of the intermediate Part, is equal to the
Square of half the given Line. Eucl. 2. Prop. 5.

THE Line AB is divided equally at C, and unequally at D.
I say, that $AD \times DB + CD^2 = AC^2$. Let AC A ——— C ——— B
or CB = a , and CD = b , then BD = $a - b$,
and $AD = a + b$, now the Rectangle of $a + b$ by $a - b$, is $aa - ab$
 $- ab - bb$ †. But $+ab - ab = 0$; therefore the Rectangle is $aa - bb$.
Consequently $AD \times DB = aa - bb$, then adding to each Part bb or
CD which is equal to bb , ‡ will make $AD \times DB + CD^2 = aa$, which
was to be proved. To add bb to $aa - bb$, is only to cancel $-bb$,
for it is evident, that $aa - bb + bb = aa$, as has been explained. §

PROP. 6. IF a right Line be divided into two equal Parts, and 22
there be another right Line joined directly to it, if the first whole
Line together with the Part added thereto considered as making
only one Line therewith, be multiplied by the added Line, together
with the Square of the whole first Line, the Rectangle or Product
will be equal to the Square of half the first Line, together with
the Line added thereto being considered as only one Line. Eucl. 2.
Prop. 6.

THE Line AB is divided into two A ——— C ——— B
equal Parts at the Point C, there is joined to it directly the right
Line BD; it must be demonstrated, that $AD \times BD + BC^2 = CD^2$.
Let AC or CB = b , then $AB = 2b$, and BC or AC = bb . Let BD
= d , then $CD = b + d$, and $2b + d = AD$, therefore $AD \times BD = 2bd$
 $+ dd$; and $AD \times BD + BC^2 = 2bd + dd + bb$; now the Square of CD
or $b + d$ is $bb + 2bd + dd$ || hence $AD \times BD + BC^2 = CD^2$; which was
to be proved, to wit, that the Rectangle of $AD \times BD$, toge-her
with the Square of AC or BC, is equal to the Square of CD.

K 2

PROP.

† Sup. n. 8.

‡ Sup. n. 8.

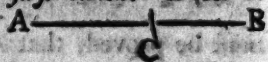
§ Sup. n. 6.

|| Sup. n. 8.

- 23 PROP. 7. IF a Line be any how divided, the Square of the whole Line together with the Square of one of its Parts, is equal to the Square of the other Part, and to twice the Rectangle contained under the whole Line, and the Part which was first taken. Eucl. 2.

Prop. 7.

THE Line AB being any how cut at



the Point C. It must be demonstrated, that $AB^2 + BC^2 = 2AB \cdot BC$

+ AC. Let $AC = b$, then $AC^2 = bb$; let $BC = d$, then $BC^2 = dd$; so

likewise since $AB = b + d$, then $AB^2 = bb + 2bd + dd$, and $AB \cdot BC = bd + dd$, doubling these two Quantities $2AB \cdot BC = 2bd + 2dd$,

and adding to them AC^2 or bb , it will become $2AB \cdot BC + AC^2 =$

$2bd + 2dd + bb$, and consequently equal to $2AB \cdot BC$; Q. E. D.

- 24 PROP. 8. IF a right Line be any how divided into two Parts, four Times the Rectangle comprehended under the whole Line, and one of the Parts, together with the Square of the other Part, is equal to the Square of the whole Line and the Part first taken considered as only one Line. Eucl. 2. Prop. 8.

THE Line AB was divided any how into two Parts at the

Point C, it must be demonstrated, that $4AB \cdot BC + AC^2$, is equal to the Square of a Line equal to $AB + BC$. Let $AC = b$, and $BC = d$, then $AB = b + d$, and $AB + BC = b + d + d$, or $b + 2d$, whose Square is $bb + 4bd + 4dd$,† which is equal to that of the Line $AB + BC$; now $AB \cdot BC = bd + dd$, since AB was supposed $= b + d$, and $CB = d$; then $4AB \cdot BC = 4bd + 4dd$, adding to each

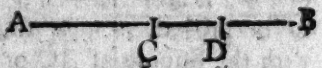
Part of AC^2 , its equal bb , it will be $AC^2 + 4AB \cdot BC = bb + 4bd + 4dd$, which is the same Value as was found for the Square of the Line $b + 2d = AB + BC$; hence the Square of $AC + 4AB \cdot BC$ is equal to the Square of the Line $AB + CB$; which was to be proved.

- 25 PROP. 9. IF a right Line be divided into two equal Parts, and into two unequal ones, the Squares of the two unequal Parts are double the Squares of half the whole Line, and intermediate Part. Eucl. 2. Prop. 9.

THE Line AB is cut into two equal Parts at C, and into two

unequal ones at D, it must be proved that $AD^2 + DB^2$, is double

$AC^2 + CD^2$. Let AC , or $CB = b$, let



$CD = d$, then $AD = b + d$, and DB

$= b - d$. Hence $AD^2 = bb + 2bd + dd$,† and $DB^2 = bb - 2bd + dd$, and

$AD^2 + DB^2 = bb + 2bd + dd - 2bd + dd$, and by reason that $+2bd - 2bd$ make

* Sup. n. 8.

† Sup. n. 8.

‡ Sup. n. 8.

make nothing, then $\overline{AD} + \overline{DB} = 2bb + 2dd$; which is the double of $\overline{AC} + \overline{CD}$; which was to be proved.

PROP. 10. IF a right Line is divided into two equal Parts, and another right Line be directly joined to it, the Square of the whole Line with the Part adjoined considered as only one Line, together with the Square of the adjoined Part, is double the Square of half the whole Line, and Square of the said half with the Part adjoined, esteemed as only one Line. Eucl. 2. Prop. 10.

THE Line AB is divided into two equal Parts at the Point C, and the Line BD is adjoined thereto, it must be demonstrated, that the Square of AB+BD, together with that of BD, is double

those of BC and CD, that is, that $\overline{AD} + \overline{BD}$ is equal to $2\overline{BC} + 2\overline{CD}$. Let AC or BC = b , then $\overline{BC} = bb$, and $\overline{AB} = 2b$; then $\overline{AB} = 4bb$. Let BD = d ; then $\overline{AD} = 2b + d$, and $\overline{AD} = 4bb + 4bd + dd$. Then the Square of AB+BD, together with the Square of BD, is equal to $4bb + 4bd + 2dd$; now the Square of BC, or bb , together with that of BC+BD, or $b+d$, is $2bb + 2bd + dd$, which is the half of $4bb + 4bd + 2dd$; which was to be proved.

PROP. 11. IN every ambliogonium Triangle, the Square of the Side which subtends the obtuse Angle, is equal to the Squares of both the other Sides, together with a double Rectangle comprehended under the Side upon which the Perpendicular, (or Height of the Triangle) falls, and its Part produced so as to meet the Perpendicular. Eucl. 2. Prop. 12.

THE Angle ACB of the ambliogonium Triangle ABC is Obtuse, and the Perpendicular AD falls upon the produced Side BC, which being so, it must be demonstrated, that the Square of AB, the Base of the obtuse Angle ACB, is equal to the Squares of both the other Sides AC and BC, together with twice the Rectangle made by the Side BC and its produced Part, comprehended between the obtuse Angle C and the Perpendicular AD; which

is expressed thus: $\overline{AB} = \overline{BC} + \overline{AC} + 2\overline{BC} \times \overline{CD}$. ADB being a right

Angle, $\dagger \overline{AB} = \overline{BD} + \overline{AD}$, and by the same reason $\overline{AC} = \overline{CD} + \overline{AD}$:

But $\overline{BD} = \overline{BC} + 2\overline{BC} \times \overline{CD} + \overline{CD}$; $\ddagger$ then substituting in the place of $\overline{BD}$; and in the place of $\overline{AD} + \overline{CD}$, the equal Quantity $\overline{AC}$, it

will be $\overline{AB} = \overline{BC} + \overline{AC} + 2\overline{BC} \times \overline{CD}$, which was to be proved.

PROP. 12. IN every oxygonium or acute-angled Triangle, the Square of one of its Sides is equal to the Squares of both the other Sides, less twice the Rectangle comprehended under one of the said Sides, and the Perpendicular

$\dagger$ B. 2. n. 142.

$\ddagger$ Sup. n. 20.

other Sides, and one of its Parts contained between the Perpendicular which cuts it, and the Angle that's opposite to the Side first taken. Eucl. 2. Prop. 13.

I suppose that the Triangle ABD is an Oxygonium, that the Perpendicular AC falls upon the Side BD, it must be demonstrated, that the Square of AB is equal to the Squares of both the other Sides AD, and BD, less twice the Rectangle made of the whole side BD, and its Part DC contained between the Perpendicular AC and the Angle D opposite to the Side AB first taken; it must

therefore be demonstrated that $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2BD \times DC$.

ACD being a right Angle, $\overline{AD}^2 = \overline{AC}^2 + \overline{DC}^2$,* taking away DC

from each Side, it becomes $\overline{AD}^2 - \overline{DC}^2 = \overline{AC}^2$; this is done as has

been taught, by cancelling DC, where it is found with the Sign +, and writing it on the other Side with the Sign —, since ACB is a

right Angle,† then $\overline{AC}^2 + \overline{BC}^2 = \overline{AB}^2$. Now $\overline{BC}^2 = \overline{BD}^2 - \overline{DC}^2$, the

Square of BD—DC is $\overline{BD}^2 - 2BD \times DC + \overline{DC}^2$, hence $\overline{BC}^2 = \overline{BD}^2 -$

$2BD \times DC + \overline{DC}^2$; therefore putting in the Place of AC and BC

the Quantities equal thereto, it will be $\overline{AB}^2 = \overline{AD}^2 - \overline{DC}^2 + \overline{BD}^2 -$

$2BD \times DC + \overline{DC}^2$: (and as $-\overline{DC}^2 + \overline{DC}^2 = 0$.) $\overline{AB}^2 = \overline{AD}^2 + \overline{BD}^2 - 2BD \times DC$; Q. E. D.

S E C T. III.

OF the Ratio's and Proportions of Lines, Surfaces and Solids.

A D V E R T I S E M E N T.

IT may be considered both what a Line is in itself, and also what it is in Proportion to others, which Proportion occasions it to be esteemed, equal, or unequal, short, or long, it is the same of Surfaces, and Solids; now one Line may be proportionable to another, and they may be compared differently, (*viz.*) by considering either the Excess of one above the other, that is, their Difference, or how often one contains the other, which make two Sorts of Proportion. Geometers consider only the latter, which they also call Ratio. which in general signifies Proportion, in Ratio's, or this latter Sort of Proportion, that we are going

* E. 2. n. 142.

† Book 2. n. 142.

to treat of, The Interpreters of Euclid formerly defined Ratio's, to be a *Habitude* of two Quantities of the same Species, compared one with another according to their Quantity. When things are not of the same kind, they cannot be compared together; for Length cannot be compared with breadth, when the Ratio of two things is mentioned, it is then their Quantity or Bigness that is compared: But the word *Habitude* was here improperly used, the Greek Term used by Euclid, which is translated *Habitude*, signifies what one thing is with respect to another; and that is what the word Ratio signifies in Geometry. But as I said before, 'tis the second Sort of Proportion that is here to be treated of; which will be explained by the following Definitions.

DEF. 1. THE Ratio of a Line to a Line, of a Plane to a Plane, of a Solid to a Solid, is the Manner by which a Line contains or is contained in that with which it is compared, that a Plane contains, or is contained in a Plane, that a Solid contains or is contained in the Solid, with which it is compared.

DIVISION shews how one Quantity is contained in another, hence to express the Proportion, or Ratio of one Line to another, as of the Line A to the Line B. We divide B by A, writing them as has been shewn one under the other, $\frac{B}{A}$ or $\frac{A}{B}$: this Expression shews or expresses the Ratio of A to B, that is, how many times, and in what manner A is in B, or what part B is of A, which is called the Exponent of the Ratio of A to B.

DEF. 2. A Ratio whose Exponent is expressible by Number, is called a Ratio of Number to Number.

THE Exponent shews either how one Quantity contains, or is contained in another, that is, (in Arithmetical terms,) it shews the Quotient of the two Quantities, when divided one by the other; if this Exponent or Quotient is a Number, as for Example. Suppose the Line B is contained six times in A, the Ratio of these two Lines A and B, is a Ratio of Number to Number.

DEF. 3. A Ratio whose Exponent cannot be expressed by Number, is called *Surd* or *Irrational*.

If there cannot be found a Number, that shews exactly how many times the Line A contains, or is contained in the Line B, the Ratio of these two Lines A and B is *surd*. That there are such Ratio's, shall be hereafter demonstrated.

DEF. 4. THE Equality of Ratio's is called *Proportion*.

IF A has the same Ratio to B, as C has to D, the four Quantities are said to be Proportional; which is expressed thus: A. B :: C. D. It has been said, that the Ratio of A to B is expressed thus $\frac{A}{B}$; hence, that of C to D is in like manner $\frac{C}{D}$.

Consequently the Proportion of these four Lines, which consists in the Equality of their Ratio's, may be also expressed thus: $\frac{A}{B} = \frac{C}{D}$.

DEF. 5.

33 DEF. 5. THE first Term of a Ratio is called Antecedent, and the other Consequent.

THE Antecedent is the Thing which is related, or compared; the Consequent is that with which the Comparison is made. Every comparison supposes two Terms: Ratio, which is a Relation or Comparison, requires therefore two Terms.

34 DEF. 6. A Proportion hath two Antecedents, and two consequents.

PROPORTION is the Equality of the Ratio's that are compared. Every Ratio supposes two Terms: Therefore Proportion requires four.

35 DEF. 7. THE same Term in a Proportion may be both Consequent and Antecedent; and when it is so, the Proportion is called Continued.

IF A is to B as B to C, this makes two equal Ratio's, and consequently makes this Proportion. $A. B :: B. C.$

Now in this Proportion B is consequent to the first Ratio, and Antecedent to the Second. Which is expressed thus, $\therefore A. B. C.$

36 DEF. 8. WHEN a continued Proportion has more than three Terms, 'tis called Progression.

IF A is to B as B to C, and B to C, as C to D, and C to D as D to E, and so on; it is called Progression, which is expressed thus, $\therefore A. B. C. D. E.$

37 DEF. 9. WHEN there are a certain Number of Quantities on each Side, for Example, three on one Side, and three on the other, if in comparing all the Six together thus, the First with the Second, and the Fourth with the Fifth, the Second with the Third, and the Fifth with the Sixth, they are in Proportion, it is called ordinate Proportion.

LET A. B. C. be on one Side, and D. E. F. on the other; if $A. B :: D. E$ and $B. C :: E. F$, they are in ordinate Proportion.

38 DEF. 10. THREE Quantities being on one Side, and three on the Other: If in comparing the First with the Second, and the Fifth with the Sixth, they are in Proportion, and that in changing the Order and comparing the Second to the Third, and the Fourth to the Fifth, they be also still in Proportion, then it is called Perturbata, or Inordinate Proportion.

LET there be $\left\{ \begin{array}{l} A. B. C. \\ 12. 3. 4. \end{array} \right.$ and $\left\{ \begin{array}{l} D. E. F. \\ 12. 6. 4. \end{array} \right.$

IF $A. B :: E. F$ and $B. C :: D. E$ this Proportion is called Perturbata or Inordinate, because it does not observe the same Order.

Def:

DEF. 11. THE first and last Terms of a Proportion, are called ³⁹ the Extremes of that Proportion; and the Second and Third, the Means.

LET the Proportion be $A. B :: C. D.$ the Extremes are A and D; and the Means are B and C.

DEF. 12. THE homologous Terms of a Proportion, are those ⁴⁰ which belong to the same Rank, or which have the same Name.

IN this Proportion $A. B :: C. D.$ the Terms A and C which are the Antecedents, are Homologous, as are also B and D which are the Consequents.

EVIDENT Propositions concerning Ratio's and Proportions.

PROP. 1. EQUAL Ratio's have equal Exponents.

THE Exponent of a Ratio shews how often one of the Terms ⁴¹ is contained in the other, therefore if they are equally contained, their Exponents are equal.

PROP. 2. EQUAL Quantities cannot be the Exponents of unequal Ratio's. ⁴²

Let X be the Exponent of the Ratio of A to B, and also of the Ratio of C to D, the two Ratio's are equal, seeing they equally contain each other.

LEMMA. WHEN the first Term of a Ratio is multiplied by the Exponent of that Ratio, it becomes equal to the second Term. ⁴³

LET A be the first Term or Antecedent of the Ratio of A to B, the Exponent of this Ratio, or which is the same Thing; the Quotient q of the Terms divided one by the other, shews how many times A is in B, or what part B is of A; consequently being taken as many times as 'tis contained therein, that is, being multiplied by q, * it ought to become equal to B the Dividend, hence $Aq=B$; which was to be proved.

SCHOLIUM.

I shall henceforward generally suppose the Antecedent to be lesser than the Consequent; thus having mentioned A and the Consequent B, I shall say that $Aq=B$, nevertheless if it were otherwise, and that the Antecedent A, were greater than the Consequent B, it would be equally true to say, that the Term A multiplied by the Exponent of A to B will always produce the Consequent B, by reason the Exponent equally shews how one contains, or is contained in the other, whether the Ratio be surd, or of Number to Number. Hence when the Antecedent A is less than the Consequent B, the exponent of that Ratio will be greater than Unit, and on the contrary lesser, (which in the Arithmetical phrase is called a Fraction)

* Sup. n. 4.

on) when the Antecedent is greater than the Consequent, but so that the Proportion be Ordinate, the first Term multiplied by the Exponent of its Ratio to the second Term, will always produce the said second Term which was divided.

44 PROP. 3. FOUR Terms will continue in Proportion howsoever changed, so long as the first Antecedent is to its Consequent, as the second Antecedent is to its Consequent. Which follows from the Definition of Proportion.

45 PROP. 4. FOUR Quantities being proportional Permutando, that is, changing them and making the Antecedents become Consequents, they still continue Proportional.

IF $A. B :: C. D$. It must be proved, that $B. A :: D. C$. To contain and be contained are reciprocal Terms, thus if A contains B, as often as C contains D, and consequently that there be an equality of Ratio's, it must be that B is contained in A, as often as D is contained in C, and that there is also an Equality of Ratio's, when a Consequence is drawn from this Proportion, it is called a Conclusion a Permutando, or Inverse Ratio.

46 PROP. 5. FOUR Quantities being Proportional, Alternando, that is, in comparing the first and second Antecedents together, and the first and second Consequents together, these four Quantities are also Proportional. Eucl. 5. Prop. 16.

LET the Proportion be $A. B :: C. D$. It must be proved, that Alternately. $A. C :: B. D$. let q be supposed to be the Exponent of the two Ratio's, * $Aq = B$, and $Cq = D$; then instead of $A. B :: C. D$. it may be wrote $A. Aq :: C. Cq$. It must then be proved, that Alternately $A. C :: Aq. Cq$. which is evident, since that the Quotient of C, divided by A is $\frac{C}{A}$, the same

as that of Cq divided by Aq , which is $\frac{Cq}{Aq}$. Now according to the Rules of Division, † q may be Cancelled it being both above and below the Line, after which there remains only $\frac{C}{A}$; therefore the two Ratio's having one and the same Exponent, they are equal.

47 PROP. 6. IF to two Terms of a Ratio, be added two other Terms of the same Ratio, thus if to the Antecedent of the First, be added the Consequent of the Second, the same Ratio will continue.

LET the Proportion be $A. B :: C. D$. It must be proved that $A+C. B+D :: A. B$. the Exponent of the Ratio's of A to B and of C to D is q . Hence $Aq = B$, † and $Cq = D$. It must then be proved that $A+C. Aq+Cq :: A. B$. The Quotient of $Aq+Cq$ divided by $A+C$, is q ; § then the two Terms $A+C$ and $Aq+Cq$ having one and the same Exponent as A and Aq , or as A and B, they have the same Ratio. ||

PROP. 7.

* Sup. n. 43.

† Sup. n. 4.

‡ Sup. n. 43.

§ Sup. n. 4.

|| Sup. n. 41.

PROP. 7. Two Terms of a Ratio being subtracted from two other Terms of the same Ratio, thus Antecedent from Antecedent, and Consequent from Consequent, the same Ratio will remain. ⁴⁸

LET the Proportion be $A : B :: C : D$. It must be proved that $A - C : B - D :: A : B$. If q , is the Exponent of these Ratio's, then $Aq = B$, and $Cq = D$. * It must therefore be proved that $A - C : Aq - Cq :: A : B$. Now the Quotient of $Aq - Cq$ divided by $A - C$, is q ; then these two Terms have the same Exponent, and therefore the same Ratio as A and B .

PROP. 8. FOUR Quantities being in Proportion, Componendo, that is, the first Antecedent, more its Consequent, is to its Consequent, as the second Antecedent more its Consequent, is to its Consequent. ⁴⁹
Eucl. 5. Prop. 18.

$A : B :: C : D$. It must be demonstrated that $A + B : B :: C + D : D$.
1. Alternately, $A : C :: B : D$. † then adding the Terms C and D , (which are in the same Ratio,) ‡ to A and B . § $A + B : C + D :: B : D$. and again Alternately $A + B : B :: C + D : D$. Q. E. D.

PROP. 9. WHEN several Quantities are in Proportion, the Sum of the Antecedents, is to that of the Consequents, as each Antecedent is to its Consequent. ⁵⁰
Eucl. 5. Prop. 12.

Suppose $A : B :: C : D :: E : F$. It must be proved that $A + C + E : B + D + F :: A : B :: C : D :: E : F$. By the preceeding Proposition $A + C : C :: B + D : D$. Alternately $A + C : B + D :: C : D$. Now the Ratio of C to D is the same, as that of E to F , therefore $A + C : B + D :: E : F$. Alternately. $A + C : E :: B + D : F$. then also according to the preceeding Proposition $A + C + E : B + D + F :: E : F$ or $A : B$, or $C : D$; for the Ratio is always the same, which was to be proved.

PROP. 10. FOUR Quantities being in Proportion, Dividendo, that is, the first Antecedent less its Consequent, is to its Consequent, as the second Antecedent less its Consequent, is to its Consequent. ⁵¹
Eucl. 5. Prop. 17.

LET $A : B :: C : D$. It must be proved that $A - B : B :: C - D : D$. since $A : B :: C : D$. then Alternately. $A : C :: B : D$. then subtracting B from A , and D from C , || $A - B : C - D :: A : C$. Now the Ratio of A to C is the same as that of B to D ; hence $A - B : C - D :: B : D$. then Alternately $A - B : B :: C - D : D$. which was to be proved.

PROP. 11. Two Quantities each having the same Ratio to a Third, are equal one to another. ⁵²
Eucl. 5. Prop. 9.

If $A : B :: C : D$, it must be that $A = C$, for let q be the Exponent of the Ratio of A to B ; it will be that of C to B , which is the same, then $Aq = B$, and $Cq = B$, * therefore $Aq = B = Cq$. these two Quantities being equal to a Third, viz. B , they are by Axiom the Third, equal to one another.

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Prop. 12

* Sup. n. 43. † Sup. n. 46. ‡ Sup. n. 46.

§ Sup. n. 47. || Sup. n. 8.

* Sup. n. 43.

PROP. 12. TWO Ratio's equal to a third Ratio, are equal to each other. Eucl. 5. Prop. 11.

If $A : B :: E : F$ and $C : D :: E : F$, it must be proved that $A : B :: C : D$. If the Exponent of the Ratio of A to B is q , and that q be also that of the Ratio of E to F , which is the same, now the Ratio of C to D is the same, as that of E to F ; therefore its Exponent is also q . The two Ratio's of A to B , and C to D : Have then one and the same Exponent; therefore they are Equal. *

PROP. 13. WHEN two Quantities are multiplied by one and the same Quantity, they continue in the same Ratio after the Multiplication, as they were in before.

If A and B are multiplied by x , it must be proved that $Ax : Bx :: A : B$. the Exponent of the Ratio of A to B is $\frac{A}{B}$, and that

of the Ratio of Ax to Bx is $\frac{Ax}{Bx}$. Now the Exponent is just the same, for having cancelled x which is both above and below the Line in $\frac{Ax}{Bx}$, there remains $\frac{A}{B}$; † therefore the Ratio's having one and the same Exponent, they are equal. †

PROP. 14. TWO Quantities being divided by a Third, the Quotients thereof are in the same Ratio as the Quantities themselves.

LET the two Quantities be B and D , divide them by x , let the Quotient of B by x be called p , and let that of D by x , be called q . It must be proved that $p : q :: B : D$. Now $px = B$, and $qx = D$; then $px : qx :: B : D$. and p and q having been multiplied by x , according to the preceeding Proposition $px : qx :: p : q$. then since $px : qx :: p : q$. it must be that $p : q :: B : D$. § Q. E. D.

PROP. 15. WHEN four Quantities are Proportional, the Product or Rectangle of the extremes, is equal to that of the Means. Eucl. 6. Prop. 16.

$A : B :: C : D$. It must be proved that $AXD = BXC$, let x be the Exponent of the two equal Ratio's; then $Ax = B$, and $Cx = D$. Hence this Proportion may be thus express'd: $A : Ax :: C : Cx$. It must then be demonstrated that $ACx = ACx$, which is evident.

COROLLARY THREE Quantities being in continued Proportion, the Product of the Extremes is equal to the Square of the mean Term. Eucl. 6. Prop. 17.

SUPPOSE $A : B :: B : C$. Since $A : B :: B : C$. then $AC = BB$, or $AC = B^2$.

PROP. 16. WHEN four Quantities are so disposed, that the Rectangle or Product of the Extremes is equal to that of the Means, they are Proportional. Eucl. 6. Prop. 16.

IN

* Sup. n. 41.

† Sup. n. 4.

‡ Sup. n. 41.

§ Sup. . 53.

In these four Lines A, B, C, D the Product AD of the Extremes is equal to BC the Product of the Means. It must be demonstrated that $A : B :: C : D$. let x be the Quotient of B divided by A, and z that of D divided by C: then $Ax = B$, and $Cz = D$. Hence I reduce the four Terms to these, A, Ax, C, Cz. According to the Supposition $AxCz = Ax \times C$, taking $A \times C$ from each Side, there will remain $x = z$; consequently the Exponent of the Ratio of A to B, is equal to that of the Ratio of C to D; therefore they are Equal: * Consequently the four Quantities are Proportional.

COROLLARY. 1. THE four Terms of a Proportion being changed, will always be ranged Proportionally, provided that the two same Terms be always either the two Extremes or the two Means.

LET the four Terms be $A : B :: C : D$. provided that A and D, for Example, be always either the two Means or the two Extremes, their Product is always equal, consequently they are Proportional, according to this Proposition.

COROLLARY 2. THE Product of the mean Terms of a Proportion being divided by the first Term, the Quotient of the Division will be the fourth Term; and if the same Product be divided by the fourth Term, the Quotient will be the first Term.

FOR the Divisor multiplied by the Quotient, makes a Product equal to the Dividend. †

DEF. THE Product of two Quantities being equal to that of two others, (they may then make a Proportion,) if the First is to the Third, as the Fourth is to the Second, the Proportion is called Reciprocal, or Inverse.

LET the two equal Rectangles be AC and EG, if $AB : BF :: FG : BC$, the Proportion is called Reciprocal, or Inverse, which begins and ends in the same Figure. If one of the Sides of the two Rectangles is longer the other is Reciprocally shorter.

PROP. 17. IF there are ever so many Quantities on one Side, and as many on the other, which being taken two and two are in the same Ratio, those which are in equal Ratio, are Proportional. Eucl. 5. Prop. 22.

LET there be three Quantities A, B, C on one Side, and three others D, E, F on the other Side. If $A : B :: D : E$ and $B : C :: E : F$. it must be demonstrated that $A : C :: D : F$. in equal Ratio. 1. Alternately, $B : F :: A : D$, and $B : E :: C : F$. † then the two Ratio's of A to D, and of C to F being equal to that of B to E, they are equal: § Consequently $A : D :: C : F$, and Alternately $A : C :: D : F$. 2. E. D.

PROP. 18. IF the first Term is to the Second as the Third is to the Fourth, and the Fifth to the Second as the Sixth to the Fourth, the Sum of the First and Fifth will be to the Second, as the Sum of the Third and Sixth is to the Fourth. Eucl. 5. Prop. 24.

LET the six Quantities be A, B, C, D, E, F; if the first A is to the second B as the third C is to the fourth D, and the fifth E

* Sup. n. 41.

† Sup. n. 4.

‡ Sup. n. 46.

§ Sup. n. 53.

E to the second B as the sixth F is to the fourth D; which is expressed thus: $A. B :: C. D$ and $E. B :: F. D$.

I say that $A+E. B :: C+F. D$; for, 1. *Alternately*, * $A. C :: B. D$ and $E. F :: B. D$; then † $A. C :: E. F$; then $A+E. C+F :: A. C. †$ and since $A. C :: B. D$; then $A+E. C+F :: B. D. §$ and *Alternately*, $A+E. B :: C+F. D. ¶. E. D.$

PROPOSITIONS concerning Proportions, which EUCLID in his fifth Book, hath expressed in a different Manner.

EUCLID calls that Quantity a *Multiple*, which is measured exactly by a lesser Quantity repeated a certain number of Times; and those Quantities *Equimultiples*, which contain the Quantities whose multiples they are, an equal number of Times, the Product of one Quantity multiplied by another, is the Multiple of that Quantity; thus Ax is the Multiple of A , but A and B having been multiplied by the same Multiplier x , the two Products Ax and Bx are *Equimultiples* of A and B . The following Propositions have been demonstrated above, the reason why they are again repeated, is their having been expressed in a manner different from that in Eucl. Book. 5.

64 PROP. 1. IF there are ever so many Quantities, Equimultiples of as many other Quantities, each to each, as one is the Multiple of one, so are the whole the Multiples of the Whole.

I express this Proposition thus: Bx and Cx being Equimultiples of B and C , I say, that the Quantity Bx is to B , and Cx to C , as $Bx+Cx$ is to $B+C$, 1. $Bx. Cx :: B. C. ||$ 2. *Alternately* $Bx. B :: Cx. C$. 3. Adding therefore the Quantities Cx and C , which are in the same Ratio, * to B and Bx it will be $Bx. B :: Bx+Cx. B+C$, and $Cx. C :: Bx+Cx. B+C$; which was to be proved.

65 PROP. 2. IF the First is the same Multiple of the Second, as the Third is of the Fourth, and the Fifth the same Multiple of the Second, as the Sixth is of the Fourth; the Sum of the First and Fifth will be the same Multiple of the Second, as the Sum of the Third and Sixth, is of the Fourth.

IF $A. B :: C. D$, and that $E. B :: F. D$. I say that $A+E. B :: C+F. D$. Since $A. B :: C. D$ and $E. B. F. D$. then *Alternately* $A. C :: B. D$, and $E. F :: B. D$; hence the Ratio of A to C is the same as that of E to F , these two Ratios are the same as that of B to D ; † therefore $A. C :: E. F$; adding E to A , and F to C , ‡ then $A+E. C+F :: B$ to D .

66 Now *Alternately*, $A+E. B :: C+F. D$; which was to be proved.

PROP. 3. IF the First is the same Multiple of the Second as the Third of the Fourth, and there be taken Equimultiples of the First and third; the Multiple of the First will be the same Multiple of the Second, as the Multiple of the Third is of the Fourth.

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* Sup. n. 46. † Sup. n. 53. ‡ Sup. n. 47.
§ Sup. n. 53. || Sup. n. 54.
¶ Sup. n. 47. † Sup. n. 53. Sup. n. 47.

A Quantity which is a Multiple contains the Quantity whose Multiple it is, therefore 'tis the same Thing as if it were said, if the First contains the Second, as many times as the Third contains the Fourth; therefore 'tis required to be demonstrated that if $A. B :: C. D$, and that E and F are Equimultiples of A and C; that is, that E contains A as often as F contains C, and that therefore $A. C :: E. F$; it must be that $B. D :: E. F$, for since $A. B :: C. D$; then Alternately, $A. C :: B. D$. Now $A. C :: E. F$. Hence the Ratio of B to D being equal to one and the same Ratio, * it is the same as that of E to F. 67

PROP. 4. IF the First is in the same Ratio to the Second as the Third is to the Fourth, the Equimultiples of the First and Third, will also bear the same Ratio to the Equimultiples of the Second and Fourth, howsoever multiplied, if they are taken as they correspond together.

THIS is as much as to say, that if $A. B :: C. D$, likewise Alternately, $A. C :: B. D$. it must be that $Ax. Cx :: Bx. Dx$. For † $Ax. Cx :: A. C$; and $Bx. Dx :: B. D$. But $A. C :: B. D$. therefore † $Ax. Cx :: Bx. Dx$. Q. E. D. 68

PROP. 5. IF one Quantity is the same Multiple of another Quantity, as the Part subtracted from one is of the Part subtracted from the other, the Remainder of one will be also the same Multiple of the Remainder of the other, as the whole was of the whole.

THIS Proposition may be expressed thus, if $A. B :: E. F$, I say that $A-E. B-F :: A. B$ for then the Quantities which are subtracted from them are in the same Ratio, therefore they will remain in the same Ratio. § 69

PROP. 6. IF two Quantities are Equimultiples of two other Quantities, and there be Equimultiples subtracted from them, the Remainders are either equal thereto, or Equimultiples thereof.

THIS Proposition I express thus. If $Ax. Bx :: Ax-C. Bx-D$. I say that $Ax-C. Bx-D :: A. B$. This is evident; for $Ax. Bx :: A. B$. || Now two Ratio's equal to a Third, are equal: * therefore if $Ax. Bx :: Ax-C. Bx-D$; it must be that $Ax-C. Bx-D :: A. B$. 70

PROP. 7. QUANTITIES are one to another as their Equimultiples are one to another, if taken as they correspond together.

$Bx. Cx :: B. C$ or $Bx. B :: Cx. C$; this has been proved. † The rest of the Proposition's of Euclid's fifth Book, will follow in their proper place.

* Sup. n. 53.

† Sup. n. 54.

‡ Sup. n. 53.

§ Sup. n. 48.

|| Sup. n. 54.

• Sup. n. 53.

† Sup. n. 54.

S E C T. IV.

OF compound Ratio's and their Properties.

DEFINITION I.

71 **R**ATIO's whose Exponents are made by the Multiplication of two or more Exponents, are called Compound.

SUPPOSE x to be the Exponent of the Ratio of A to B, and that $x=xy$, then the Ratio of A to B is said to be compounded of the two Ratio's, whose Exponents are x and y .

72 DEF. 2. A Ratio compounded of two equal Ratio's is called a duple Ratio.

IF the Exponent of the Ratio of A to B is made by x multiplied by x , that is, if xx be the Exponent of that Ratio, it is a duple Ratio.

73 COROLLARY HENCE a Ratio whose Exponent is a Square, is a duple Ratio.

LET xx , or x^2 , be the Exponent of the Ratio of A to B; it is composed of x multiplied by x , and therefore of two equal Exponents; it is therefore the Exponent of a duple Ratio.

74 DEF. 3. A Ratio compounded of three equal Ratio's is called a triple Ratio.

75 COROLLARY HENCE a Ratio whose Exponent is a Cube, is a triple Ratio.

IF the Cube xxx , or x^3 , is the Exponent of the Ratio of A to B, 'tis a triple Ratio, for its Exponent is made by the Multiplication of these three equal Quantities x , x , x .

76 THEOREM 1 SEVERAL Quantities being in Progression, the Ratio of the First to the Last, is compounded of the Ratio's of all the Quantities which are between the two Extremes.

LET the Quantities be A, B, C, D, E, F, &c. It must be demonstrated that the Ratio of A to C is compounded of those of A to B, and of B to C. let x be the Exponent of A to B, then $Ax=B$; * that of B to C is z ; then by the same Reason Bz , or $Axz=C$. Hence the three Quantities A, B, C. may be changed into these A, Ax, Axz. Divide Axz by A; the Quotient xz is the Exponent of the Ratio of A to Axz, which is made by the Exponents of those Ratio's multiplied one by the other; wherefore by the first Definition, the Ratio of A to C is compounded of those of A to B, and of B to C. By this method I can demonstrate, that the Ratio of A to F, is compounded of all the Ratio's of the intervening Quantities.

77 THEOREM 2. THE Ratio of two Planes is composed of the Ratio's which the Sides of One have to the Sides of the Other, of the Breadth to the Breadth, of the Length to the Length.

Let

* Sup. n. 43.

LET the two Planes be ab and cd , it must be demonstrated that their Ratio is compounded of that of a to c and of b to d , let z be the Exponent of the Ratio of a to c ; then $az = c$. Let x be that of b to d , then $bx = d$, and $axbx = cd$. Dividing $axbx$ by ab , the Quotient will be zx composed of z and x the Exponents of the Ratio's of a to c , and of b to d : Therefore the two Planes are in a Ratio compounded of those of a to c and of b to d , that is, of their Sides Q. E. D.

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It must be remembered that it is here always supposed that the Planes and Solids are all Rectangles, therefore it was that this Remark was made before.

THEOREM 3. THE Ratio of one Solid to another, is compounded of the Ratio which the three Sides of one have to the three Sides of the other.

LET the two Solids be abc and def ; it must be demonstrated that their Ratio is compounded of these three Ratio's $\frac{a}{d}$, $\frac{b}{e}$, $\frac{c}{f}$, the Ratio of abc to def , may be expressed thus $\frac{abc}{def}$. Now this Exponent is composed of the three Exponents of those very Ratio's, of which it is required to prove that the Ratio of these two Solids is compounded, wherefore according to the first Definition, these two Solids are one to another in a compound Ratio to that of their Sides.

THEOREM 4. WHEN four Quantities are proportional, the Product of the Antecedents is to that of the Consequents, in duple Ratio to that of each Antecedent to its Consequent, or as the Square of each Antecedent is to the Square of its Consequent.

LET $a. b :: c. d$; the Product ac of the Antecedents, according to the preceding Proposition, is to bd that of the Consequents, in Ratio compounded of those of a to b and of c to d , which being equal, the Ratio is duple. * The Square aa of the Antecedent a is to bb , the Square of the consequent b , in Ratio compounded of the Ratio's of a to b and of a to b ; and consequently the double of these two Ratio's: Now these Ratio's are the same as the two Ratio's of a to b and c to d ; consequently the product ac is to the Product bd , as the Square aa is to the Square bb .

COROLLARY. SIMILAR Planes, that is, those whose Sides are proportional, are one to another in Ratio duple that of the Sides of one to the Sides of the other.

THEIR Ratio is composed of the Ratio's of the Sides of one to the Sides of the other; † these two Ratio's are equal, 'tis therefore

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* Sup. n. 72.

† Sup. n. 77.

fore a duple Ratio. * Hence all Squares being similar Planes, they are in Ratio duple that of their Sides.

- 81 THEOREM 5. WHEN six Quantities are proportional, the Product of the three Antecedents is to that of the three Consequents, in triple Ratio of each Antecedent to its Consequent, or as the Cube of each Antecedent to the Cube of its Consequent.

LET there be $a. b. c :: d. e. f.$ the Product abc is to the Product def , in Ratio compounded of those of a to d , of b to e , and of c to f ; † Now these three Ratio's are equal: This compound Ratio, is therefore triple. ‡ The Ratio aaa the Cube of a is to ddd the Cube of d in triple Ratio of that of a to d ; now 'tis the same Ratio as those of a to d , of b to e , and of c to f , consequently the Products we are speaking of are one to another, as the Cube of each Antecedent is to the Cube of its Consequent.

- 82 COROLLARY. SIMILAR Solids, that is, whose Sides are proportional, are in Ratio triple of the Sides of one to the Sides of the other.

THE Sides of two similar Solids are six proportional Quantities; therefore the Ratio of one to the other is triple. Hence all Cubes being similar Solids, are in Ratio triple those of their Sides.

- 83 THEOREM 6. IN a Geometrical Progression, the Ratio of two Terms between which are two other intervening ones, is duple, and if there are three intervening ones, it is Triple.

LET $A. B. C. D.$ the Ratio of A to C is composed or equal to a Ratio composed of that of A to B , together with that of B to C , § Now these Terms being in Progression, the two Ratio's are equal; therefore the Ratio which they compose is a duple Ratio || In the same Manner 'tis demonstrated, that the Ratio of a to d is triple. *

- 84 THEOREM 7. WHEN Quantities are proportional, their Squares are proportional.

IF $a. b :: c. d.$ I say $aa. bb :: cc. dd.$ these Squares are in Ratio duple that of their Sides, † that is, in duple Ratio of a to b , and of c to d , and as these two Ratio's are equal by Supposition, those which they compose are also equal, the Ratio of aa to bb , is then equal to that of cc to dd ; therefore the Squares are proportional,

- 85 THEOREM 8. WHEN Quantities are proportional, their Cubes are proportional.

IF $a. b :: c. d.$ I say that $aaa. bbb :: ccc. ddd.$ for the Ratio of a^3 to b^3 and that of c^3 to d^3 are triple the same Ratio's, ‡ therefore they are equal.

- 86 THEOREM 9. THREE Quantities being in continued Proportion, the Square of the first is to that of the second, as the first is to the third.

LET

* Sup. n. 72. † Sup. n. 78. ‡ Sup. n. 74.

§ Sup. n. 76. || Sup. n. 72.

* Sup. n. 74. † Sup. n. 80. ‡ Sup. n. 82.

Let there be $\therefore a. b. c.$ I say that $a^2. b^2 :: a. c.$ the Ratio of a to c , is compounded of the two Ratio's of a to b and of b to c , † these two Ratio's are equal; therefore the Ratio of a to c is duple, || now the Ratio of a^2 to b^2 is also duple the same Ratio's; † then $a^2. b^2 :: a. c.$

THEOR. 10. FOUR Quantities being in continued Proportion, the Cube of the First is to the Cube of the Second, as the First is to the Fourth.

Let them be $\therefore b. c. d. f.$ The Ratio of b to f is compounded of the three intervening Ratio's, * and these three Ratio's being the same, the Ratio of b to f is triple, † now the Cube b^3 is to the Cube f^3 in a Ratio triple the same Ratio: † Therefore $b^3. f^3 :: b. f.$

THEOR. 11. IF three Quantities and as many others taken two and two are in the same Ratio, and in perturbate Proportion, the Quantities that are in equal Ratio are Proportional. Eucl. 5. Prop. 23.

THREE Quantities A, B, C , and others D, E, F taken two and two, are in the same Ratio, but the Proportion is perturbate, that is, that $A. B :: E. F$, and $B. C :: D. E$, it must be demonstrated that $A. C :: D. F$. The Ratio of A to C is compounded of those of A to B , and of B to C , also that of D to F is compounded of those of D to E , and of E to F , now these two Ratio's are compounded of equal Ratio's; for $A. B :: E. F$, and $B. C :: D. E$. Consequently the compounding Quantities being equal, the Quantities composed thereof are equal, hence $A. C :: D. F$. Q. E. D.

SECT. V.

OF the Comparison of Ratio's.

PROBLEM.

TO reduce two different Ratio's so as to have the same Consequent.

Let the two Ratio's be $\frac{a}{b}$ and $\frac{c}{d}$, to reduce them to the same Consequent, I multiply the Antecedent and Consequent of the First by the Consequent of the Second; which produces $\frac{ad}{bd}$; likewise multiply the Antecedent of the second Ratio, and its

† Sup. n. 76.

|| Sup. n. 72.

† Sup. n. 30.

Conse-

* Sup. n. 76.

† Sup. n. 74.

† Sup. n. 82.

Consequent by the Consequent of the First; which produces $\frac{bc}{bd}$; hence the two proposed Ratio's are reduced to these $\frac{ad}{bd}$ and $\frac{bc}{bd}$. these two Ratio's have the same Consequent, nevertheless there is the same Proportion between the Antecedent and its Consequent, as there was before. §

90 THEOREM 1. Two Ratio's having one and the same Consequent, are one to another as their Antecedents.

LET the two Ratio's be those of A to X, and of B to X, which have one and the same Consequent X, it must be demonstrated that they are one to another, as the Antecedents A and B, divide A and B by X, and the First will be expressed thus $\frac{A}{X}$, and the Second $\frac{B}{X}$, now || A and B being divided by one and the same Divisor X, the Quotients $\frac{A}{X}$ and $\frac{B}{X}$ are one to another as A and B; Q. E. D.

91 THEOR. 2. IF two Quantities are unequal, the Greatest has a greater Ratio to one and the same third Quantity, than the Lesser, and the Ratio of that third Quantity to the Lesser is greater than its Ratio to the Greatest. Eucl. 5. Prop. 8.

LET the two unequal Quantities be A and B, the Greatest is A; and chuse at pleasure any third Quantity as C, it must be demonstrated. 1. That the Ratio of A to C is greater than that of B to C, I express the two Ratio's thus $\frac{A}{C}$ and $\frac{B}{C}$, or $\frac{AC}{C}$ they have one and the same Consequent; therefore according to the preceding Proposition, they are one to another as A and B: Now A is greater than B; therefore $\frac{A}{C} > \frac{B}{C}$. 2. It must be demonstrated that the Ratio of C to B is greater than that of C to A, that $\frac{C}{B} > \frac{C}{A}$. Reduce these two Ratio's to the same Consequent, † thus $\frac{ACBC}{AB}$, the Ratio $\frac{AC}{AB}$ is then the same as $\frac{C}{B}$; and the Ratio $\frac{BC}{AB}$, the same as $\frac{C}{A}$; ‡ now these two Ratio's $\frac{AC}{AB}$ and $\frac{BC}{AB}$ are one to another, § as A is to B; but according to the Proposition, A is greater than B; therefore the Ratio of C to B is greater, than that of C to A; Q. E. D.

Theorem

§ Sup. n. 54. || Sup. n. 55. † Sup. n. 89.
‡ Sup. n. 54. § Sup. n. 90 and 54.

THEOR. 3. OF two unequal Quantities, that is the Greatest, which hath the greater Ratio to a Third, and on the Contrary, that is the Least, to which the said third Quantity has the greater Ratio. Eucl. 5. Prop. 10.

LET A and B be the two unequal Quantities, A having a greater Ratio to C, than that of B to C; thus $\frac{A}{C} > \frac{B}{C}$, and $\frac{C}{B} > \frac{C}{A}$.

I say that $A > B$; which must be demonstrated. 1. It is then supposed $\frac{A}{C} > \frac{B}{C}$, now * these two Ratio's are one to another as A

and B; it must therefore be that $A > B$. 2. If $\frac{C}{B} > \frac{C}{A}$ It must

be demonstrated that B is lesser than A, having reduced these two

Ratio's to the same Consequent $\frac{C}{B}$ to $\frac{CA}{BA}$, and $\frac{C}{A}$ to $\frac{CB}{BA}$, it must

be according to what was supposed, that $\frac{CA}{BA} > \frac{CB}{BA}$, now these

two Ratio's are one to another as A and B; † it must then be that A is the Greater and B the Lesser.

PROP. 5. IF the First is to the Second as the Third to the Fourth, 93 and that the Third hath a greater Ratio to the Fourth, than the Fifth hath to the Sixth; the First will also have a greater Ratio to the Second, than the Fifth hath to the Sixth. Eucl. 5. Prop. 13.

LET the six Quantities be these A, B, C, D, E, F, the four First of which are in Proportion A. B::C. D, or $\frac{A}{B} = \frac{C}{D}$: It

must be demonstrated that if $\frac{C}{D} > \frac{E}{F}$, the Ratio $\frac{A}{B}$ will also be

greater, than the Ratio $\frac{E}{F}$; since $\frac{A}{B}$ and $\frac{C}{D}$ are one and the same

Ratio, $\frac{C}{D}$ being greater than $\frac{E}{F}$, It must be that $\frac{A}{B}$ is also great-

er than $\frac{E}{F}$.

PROP. 6. IF the First is to the Second as the Third is to the Fourth, and that the First be greater than the Third; the Second will 94 also greater than the Fourth; and if equal, equal; if lesser, less. Eucl. 5. Prop. 14.

LET the Proportion be A. B::C. D. It must be demonstrated that if $A = C$, also $B = D$; and that if $A > C$, likewise $B > D$;

and

* Sup. n. 90.

† Sup. n. 90.

and in short if $A < C$, so likewise $B < D$. Alternately, $A : C :: B : D$; which cannot be, unless as $A = C$, that likewise B be also equal to D ; or if A being greater than C , that B is likewise greater than D ; or if A being lesser than C , the Term B , be also lesser than D .

95 PROP. 7. If three Quantities on one Side and three on the Other, being taken two and two, and that in an equal Ratio the First is greater than the Third; the Fourth will be also greater than the Sixth; and if Equal, Equal; if Lesser, Less. Eucl. 5. Prop. 2.

LET A, B, C . be on one Side, and D, E, F . on the Other, let $A : B :: D : E$ and $B : C :: E : F$. If $A > C$, I say that $D > F$. If $A < C$, that $D < F$. If $A = C$, that $D = F$. otherwise A is not to C , as D is to F , (as was supposed) if it be either greater or lesser; for two unequal Quantities cannot contain a third or be contained therein, equally alike.

PROP. 8. If three Quantities on one Side, and three on the Other, taken two and two, are in the same Ratio, they being in perturbate Proportion, and that in equal Ratio the First is greater than the Third: The Fourth will be also greater than the Sixth; and if Equal, Equal; if Lesser, Less. Eucl. 5. Prop. 21.

LET there be A, B, C and D, E, F . If $A : B :: E : F$, and $B : C :: D : E$, and that A is greater than C ; I say that D is greater than F ; if Equal, Equal; if Lesser, Less: for C having been supposed lesser than A ; the Ratio of C to B is lesser, than that of A to B : * then since $B : C :: D : E$, and by Permutation, $C : B :: E : D$, the Ratio of E to D is less than that of A to B , or than that of E to F , which is the same: D is therefore greater than F ; † the rest is easy.

PROP. 9. If four Quantities are in Proportion, the Sum of the Greatest and Least, is greater than the Sum of the Other two. Eucl. 5. Prop. 25.

LET the four proportional Quantities be $A, B :: C, D$. I suppose A the greatest and D the least, and I say that $A + C > B + D$. Let X be the excess of A above B , and let Z be the excess of C above D ; hence $A = X + B$, and $C = Z + D$, in the Place of A and C I put what they are equal to, and I have this Proportion $X + B : B :: X + D : D$, which is the same as the Preceding one; it must then be demonstrated that $X + B + D > B + Z + D$. Subtract from each part the Quantity $B + D$, hence it remains only to demonstrate that $X > Z$. Seeing X is the difference between A and B , and Z that of C to D , $A - B = X$, and $C - D = Z$: Now $A - B : B :: C - D : D$. † then putting X and Z in the Place of their equal Quantities, $A - B$ and $C - D$, it will be $X : B :: Z : D$, or Alternately § $X : Z :: B : D$, and $B : D :: A$.

* Sup. n. 91.

† Sup. n. 92.

‡ Sup. n. 51.

§ Sup. n. 46.

A. C. according to the Supposition, hence A being greater than C, also $X \angle Z$; which remained to be proved.

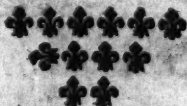
PROP. 10. QUANTITIES composed of equal and unequal ones, are one to another as the unequal ones.

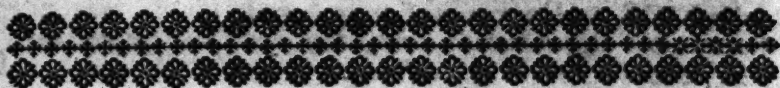
LET the Quantities be these two ab and ac , they may be imagined as compounded of b and c multiplied by a ; hence ab , $ac :: b. c$, || so likewise abd , $acd :: b. c$.

COROLLARY HENCE Triangles and Parallelograms having the same Altitude, are one to another as their Bases. Eucl. 6. Prop. 1:

THE Bigness of Triangles and Parallelograms * depends upon their Heights and Bases; therefore if their Heights are the same, they are to each other as their Bases.

|| Sup. n. 54. * E. 2. n. 130 and 135.





B O O K. IV.

S E C T. I.

Of the Ratio's and Proportions of Lines,

ACCORDING to the Idea which has been given of Ratio's and Proportions, it is evident, that in order to demonstrate that four Lines are in Proportion, it must be shewn that if they are divided into two, three, or as many parts you please; if each part of the First is Equal to each part of the Second, each part of the Third will also be Equal to each part of the Fourth; if the parts of the First be greater than those of the Second, those of the Third will also be greater than those of the Fourth; if Lesser, Less; and that if each part of the First be not contained exactly any certain number of times in the Second, each part of the Third will not be exactly contained in the Fourth, in Fine, that if there be Excess or Defect, in dividing the first and second One by the Other, there will also be Excess or Defect in dividing the Third and the Fourth one by the Other. This is the most natural Method, seeing Ratio is only an expression, which shews how one Quantity contains, or is contained in another; and it cannot be more plainly proved that a first Line is contained in a Second, as a Third is in a Fourth, than by proving it of four Lines.

¹ DEFINITION. A PARALLEL Space is that which is between two Parallel Lines.

X and Z being two parallel Lines, the Space which is between them is called a parallel Space.

² LEMMA 1. If a parallel Space, or the Perpendicular which measures it, be divided by parallel Lines, the oblique Lines contained in that Space will be divided into as many parts, as the Perpendicular.

THE Perpendicular D C is the Measure of the parallel Space between X and Z, the Line A B is Oblique, and contained in that Space; divide D C into three parts by drawing the two Parallels E and F; I say that the two Parallels do also cut the oblique Line A B into three Parts. The two Parallels E and F divide the whole parallel Space, which is between X and Z, into three parallel Spaces, thro' which the oblique Line A B passeth,

as well as DC, it is therefore divided into as many parts, as the Perpendicular DC; *which was to be proved.*

LEMMA. 2. OBLIQUE Lines, which make the same Angles in equal Parallel Spaces, are equal, and equally Oblique. 3

LET Z and X be two equal parallel Spaces, in which the oblique Lines BC and EF make the equal Angles BCH and EFG; I say that the two Lines are equal, and equally Oblique. From the Points B and E, draw Perpendicularly the Lines BH and EG, which are equal; * therefore the Triangles BCH, and EFG are Rectangled and entirely equal; † and therefore BC and EF are equal Lines; as also CH and FG; consequently ‡ BC and EF are equally Oblique; *which was to be proved.*

LEMMA. 3. OBLIQUE Lines which make the same Angles in unequal parallel Spaces, are unequal; longer, as the Space is greater, and shorter, as the Space is lesser. 4

THE Oblique Lines BC and FG make the same Angles, in the Spaces X and Z; the Perpendicular AB is longer than EF; therefore the Space X is greater than the Space Z. *It must be demonstrated, that the Oblique Line FG, is shorter than the Oblique Line BC.* Upon BA let there be taken the part B equal to EF, and thro' H let there be drawn a Parallel to the Base AC; the two Triangles ABC and EFG, being right Angled, and the Angle BCA being equal to FGE, by the Supposition they are Equiangular; § the Angle BKH is equal to BCA, and BHK to BAC; || therefore the Triangle BKH is Equiangular with BAC, and likewise with FGE. FE is supposed equal to BH; therefore FG = BK: * Now BK is part of BC; then FG its equal is also part of BC, and consequently lesser, Q. E. D.

THEOREM. 1. HAVING divided a parallel Space by two or more Parallels, the Perpendicular of that Space, and the oblique Line, are cut Proportionally. 5

1. By Lemma the first, † the oblique Line is cut into as many parts as the Perpendicular, if for Example the Perpendicular is cut into ten parts, the Oblique Line will also be cut into ten parts. 2. By the second Lemma, ¶ If the Parts of the Perpendicular are equal one to another, those of the Oblique Line are also equal one to another, for the Oblique Lines make the same Angles with the Parallels; ‡ therefore they are equal and equally Oblique, according to this Lemma. If the ten Parts into which the Perpendicular is divided are all equal, the ten Parts of the Oblique Line are then also all equal. 3. According to Lemma 3. If the Parts of the Perpendicular are unequal, those of the Oblique Line are also unequal. Hence it follows, that if there be ten equal

N

Parts

* B. 1. n. 65.

† B. 2. n. 96.

‡ B. 1. n. 55.

§ B. 2. n. 80.

|| B. 2. n. 27.

* B. 2. n. 96.

† Sup. n. 2.

¶ Sup. n. 3.

‡ B. 2. n. 27.

Parts taken from the Perpendicular, and that there remain one Part which is either lesser or greater, the Oblique Line will be so divided, that after ten equal Parts be taken therefrom, there will remain one which is lesser, if the Remainder of the Perpendicular was lesser, and greater, if the Remainder of the Perpendicular was greater, *as was proved in Lemma 3.* Wherefore as the whole is contain'd, or as the whole contains, so the Parts are contained, or do contain, therefore according to the Nature of Proportion, the two Lines we are treating of are cut Proportionally.

- 6 THEOR. 2. IF several Oblique Lines are in one and the same parallel Space, and the Space be cut by a parallel Line, the Lines will be divided Proportionally.

THE Oblique Lines EF and MN are between two Parallels, between which AC is Perpendicular. This Space is divided by a Parallel Z: then by the preceeding Theorem MN. AC::MD. AB. and likewise EF. AC::EG. AB, and Alternately MN. MD::AC. AB::EF. EG. therefore MN. MD::EF. EG, and by Permutation, MN. EF::MD. EG; which was to be proved.

- 7 THEOR. 3. OBLIQUE Lines which make the same Angles in different parallel Spaces, are one to another as those Spaces.

THE Oblique Lines BC and FG form the equal Angles BCA and FGE; consequently if AB is equal to EF, by the second Lemma * BC=FG. If AB is greater than EF, by the third Lemma † BC is greater than FG. If AB for Example is triple EF, then BC is triple FG; for supposing that BA is divided into three equal Parts; then by the first Lemma ‡ BC is also divided into three Parts, which by the second Lemma, § are each equal to GF, for these parts make the same Angles; ¶ therefore they are equally Oblique: Therefore BC, is triple FG, Q. E. D. Thus it is demonstrated that the Oblique Line FG is the same Part of the Oblique BC, as EF is of AB; or that FG is also contained in BC, as often as EF is contained in AB. If AB is equal to, or contains EF, one or more times, more some remainder, then BC is likewise equal to, or contains FG, exactly one or more times, more the same Remainder; therefore the Lines are equally Oblique, &c. Q. E. D.

* Sup. n. 3.

† Sup. n. 4.

‡ Sup. n. 2.

§ Sup. n. 3.

¶ B. 2, n. 27.

S E C T. II.

OF the Ratio's and Proportions which the Sides of Triangles bear to each other.

DEFINITION I.

TWO Triangles are Similar, when their Sides make equal Angles, or which are Equiangular.

DEF. 2. IN comparing similar Triangles, the Sides which are opposite to equal Angles, are called Homologous Sides.

X and Z are similar or equiangular Triangles, each of their Sides as AB and DE which are opposite to the equal Angles C and F, are called Homologous, that is, proportional; we shall shew, that this is a proper Term thereto.

THEOR. 2. THE Sides of two similar Triangles are Proportional. Eucl. 6. Prop. 4.

THRO' the Vertex of the two Triangles ABC and DEF draw Lines Parallel to their Bases, and from their Vertices let fall upon the Bases the Perpendiculars X and Z, the Angles ABC and DEF are equal, the Oblique Lines AC and DF form also the same Angles: Therefore $AB : DE :: X : Z :: AC : DF$. * Hence $AB : DE :: AC : DF$; by drawing Lines thro' D and E Parallel to the Sides AC and DF, it will be likewise demonstrated that $AB : DE :: BC : EF$, and that two similar Triangles have therefore all their Sides proportional.

THIS for this Reason that the corresponding Sides of similar Figures are called Homologous, because they are Proportional one to another, or have the same Ratio, which the Word Homologous implies.

THEOR. 2. TWO Triangles similar to a Third are similar one to another. Eucl. 6. Prop. 21.

LET A, B, C, be three Triangles, if A and C are similar to B, the Angles of B are equal to those of A and C; then the Angles of A and of C are also equal one to the other, † therefore according to the first Definition, ‡ A and C are Similar to each other.

THEOR. 3. IF the Sides of two Triangles are Proportional, they are Similar. Eucl. 6. Prop. 5.

THE Sides of the two Triangles ABC and DEF are such, that $AB : BC :: FE : ED$, and $AC : AB :: DF : EF$; I say that these two Triangles are Similar. Upon EF describe the Angle GEF equal to ABC, and the Angle EFG equal to BAC; therefore EGF is equal to ACB; § consequently EFG and ABC are two similar

* Sup. n. 7.

† E. 3. n. 52.

‡ Sup. n. 3.

§ B. 2. n. 80.

lar Triangles; * it remains then only to shew, that the two Triangles EFG and EFD are equal; and that as EFD, is the same as EFG, it is similar to ABC. Seeing EFG and ABC are Similar, then $AB:BC::EF:EG$, by the Supposition, $AB:BC::EF:ED$; then EG and ED, which have one and the same Ratio to EF, are equal. † The same way 'tis demonstrated that all the Sides of EFG, are equal to those of EFD; which was to be proved.

13 THEOR. 4. Two Triangles are similar, when they have one Angle equal, and the Sides containing that Angle Proportional. Eucl. 6. Prop. 6.

THE Angle FED is equal to ABC, and $AB:BC::FE:ED$; I say that ABC and FED are entirely Similar, to prove it. Make the Triangle EFG Similar to ABC, the Triangles EFG and DEF are equal; for $AB:BC::EF:EG::EF:ED$, as EG and ED have one and the same Ratio to EF; they are therefore equal. ‡ Now since the Angle DEF is supposed equal to the Angle ABC, it is then also equal to FEG, which is made equal to ABC; therefore the two Triangles DEF and GEF; having two Sides equal, EG to ED, and EF to EF, and the Angles FED and GEF which these Sides contain equal, they are equal; § then since DEF and CBA are Similar to EGF, they are Similar one to another. ||

14 THEOREM 5. If two Triangles have one Angle of one equal to one Angle of the other, and the Sides of another Angle proportional, and the third Angle of each be of the same Sort, that is, Acute, Right, or Obtuse, then are the two Triangles Equiangular; and the Angles, whose Sides are Proportional, are equal. Eucl. 6. Prop. 7.

LET the two Triangles be ABC and DEF, the Angle A is equal to the Angle D, and the Sides which contain one of the other Angles are Proportional; I say, that if C and F are of the same Sort, the two Triangles are Equiangular; and the Angles, whose Sides are Proportional, are equal. 1. Let C and F be Acute; I say that the Triangles are Equiangular, (viz.) That the Angles B and E are equal, as also E and F. If B is equal to E, according to the preceeding Proposition, the Triangles are Equiangular. But if B is greater than E, let ABG be made equal to DEF, * whose third Angle AGB is equal to the third Angle F, † and therefore Acute as that is, and ABG and DEF are Equiangular, and Similar; therefore $AB:BG::DE:EF$. now it is supposed that $DE:EF::AB:BC$. accordingly $AB:BG::DE:EF::AB:BC$: then BC and BG having one and the same Ratio with AB, they are equal; ‡ therefore the Triangle GBC is Iso-

celes

* Sup. n. 8.

† B. 3. n. 52.

‡ B. 2. n. 52.

§ B. 2. n. 98.

|| Sup. n. 11.

* B. 2. n. 29.

† B. 2. n. 76.

‡ B. 3. n. 52.

eeles, and Consequently the Angles BCG and BGC are Acute,* and Consequently BGA is greater than a right Angle: † But it hath been demonstrated that the Angle AGB is equal to the Angle F; which is supposed Acute, therefore it is at the same Time both greater and lesser than a right Angle; *which is absurd.* that if C and F are not Acute, then BGC equal to C is not Acute; which is false ‡ consequently the Angles ABC and E must be equal, hence ABC and DEF are entirely Equiangular, § therefore if in the two Triangles, &c.

THEOREM 6. If two similar Triangles have one Point common, and the Homologous Sides Parallel, the two other Sides meet directly, that is, they are in a direct Line with each other. Eucl. 6. Prop. 32. ¹⁵

LET the two similar Triangles be ABC, DCE, which have a Point common (*viz.*) C, the Sides AB and DC are Parallel, as are also AC and DE; I say that BC+CE is a right Line. Seeing AB and DC are Parallel, the Angle BAC=ACD, || and the Angle ABC=DCE per Supposition; then the three Angles ACB, ACD, DCE, are equal to the three Angles ACB, CAB, ABC of the Triangle ABC, which together make two right Angles; * therefore BCE is a right Line. †

THEOREM 7. WHEN two sides of a Triangle are cut by a Line Parallel to the Side which subtends the Angle which they contain, they are cut Proportionally. Eucl. 6. Prop. 2. ¹⁶

LET the Triangle be ABC, draw EF Parallel to BC. I say that AE. EB::AF. FC. the Triangle AEF is similar to the Triangle ABC, seeing they are Equiangular; † for besides that the Angle A is common, the Angle AEF=ABC, and the Angle AFE=ACB: § then AB. AE::AC. AF, || and dividedly, * AB—AE. AE::AC—AF. AF, now AB—AE=EB, and AC—AF=FC; then EB. AE::FC. AF, and Permutatively, AE. EB::AF. FC.

THEOR. 8. If the Angle of a Triangle is cut equally in two ¹⁷ by a right Line, which also cuts the Base, the Segments of the Base are one to another as the two other Sides, and if they be so, the Line drawn from the Vertex to the Base divides the Angle equally in two. Eucl. 6. Prop. 3.

LET the Triangle be ABD, the Line BC drawn from the Vertex, cuts the Angle B equally in two, 'tis required to be proved, 1. That AB. BD::AC. CD. let there be drawn DE Parallel to BC, and produce the Side AB, until it meet the Parallel DE, the Angles ABC and AED are equal, † and by the same Reason CBD and BDE are equal. ‡ it is supposed that ABC and CBD are equal: therefore CBD and BED are also equal, and so consequently

* B. 2. n. 84. † B. 2. n. 17. ‡ B. 2. n. 84.

§ B. 2. n. 80. || B. 2. n. 25.

* B. 2. n. 75. † B. 2. n. 80. ‡ Sup. n. 8.

§ B. 1. n. 27. || Sup. n. 10.

* B. 3. n. 51. † B. 2. n. 27. ‡ B. 2. n. 23.

quently are BDE and BED. therefore the Triangle DBE, being Isocles, $BD=BE$. now $AB:AC::BE$ (or BD .) CD . § then $AB:AC::BD:CD$. and Alternately $AB:BD::AC:CD$. 2. It must be proved that if all this be so, then BC divides ABD equally in two, since Supposing as above DE Parallel to BC , and AB produced to E , then $AB:BE::AC:CD$, * and by the Hypothesis $AB:BD::AC:CD$. then $BE=BD$, † and the Triangle DBE is Isocles, ‡ and hath the Angle $BED=BDE$; § but on account of the Parallels BC, ED , the Angles ABC, BED or its equal BDE are equal || as are also CBD, BDE , * therefore the Angle $ABC=CBD$; which was to be proved.

18 DEF. 1. THOSE Lines which make the same Angles with the Lines which they cross but which are taken from the other Side are called Antiparallels.

PARALLEL Lines make with the Lines which they cross, the same Angles taken on the same Side, □ thus if $AFE=ABC$ the Lines FE and BC are Parallel. But if $AFE=ACB$ the Lines are Antiparallels.

19 THEOR. 9. WHEN the Sides of a Triangle are cut by Lines Antiparallel to its Base, they are cut in reciprocal Proportion.

FE is Antiparallel to BC ; therefore $AFE=ACB$, and $AEF=ABC$. the Triangle AEF hath then the same Angles as ABC , and consequently they are Similar, and their Sides are Proportional, △ but their homologous Sides have not the same Position, for AB is not Homologous with AF , but with AE ; therefore the Sides AB and AC are not divided in direct Proportion. AB is not to AF as AC is to AE , so that of these four Quantities, the First is to the Fourth as the Third is to the Second, $AB:AE::AC:AF$.

20 PROBLEM 1. To divide a right Line in the same Manner as one that is already divided. Eucl. 6. Prop. 10.

THE Line AD is divided into the three Parts AB, BC, CD , 'tis required to divide the Line AG into three Parts proportional to those of AD ; join AG to AD ; so as to make any Angle you will, then draw a right Line thro' the Points G and D , and draw Lines Parallel thereto thro' the Points B and C of the divided Line; which Parallels divide AG in three parts, which are Proportional to those of AD : for † $AD:AC::AG:AF$, and $AC:AB::AF:AE$; therefore what was required is done.

21 PROBLEM 2. To divide a right Line into as many equal Parts you please.

THE given Line is X , required to be divided into three Parts. Join thereto an indeterminate Line, making therewith any Angle at pleasure as ZAX , with any extent of the Compasses, and one

Foot

§ Sup. n. 16. * Sup. n. 16. † B. 3. n. 52.
 ‡ B. 2. n. 57. § B. 2. n. 83.
 || B. 2. n. 27. * B. 2. n. 25. □ B. 2. n. 27.
 △ Sup. n. 10. † Sup. n. 16.

Foot in A, mark upon Z with that extent three equal Parts, then from B the end of the three Parts, draw a Line to the Point C, the End of X, and thro' the Divisions of Z, draw Parallels thereto, then according to what was demonstrated in the preceding Problem, AC will be divided like AB, that is, into three Parts.

PROB. 3. To cut off from a given right Line any Part required. 22
Eucl. 6. Prop. 9.

If from AC, *fig. preced.* It was required to cut off the third Part, you need only divide AC into three Parts, according to the preceding Problem, and take from AC one of these three Parts.

PROB. 4. Two Lines being given, to find a third in Proportion thereto. Eucl. 6. Prop. 11.

THE first Line is AB, the Second BC, join them together so as they may make only one right Line: then take AD equal to BC, which join to AB making any Angle therewith at Pleasure; from D draw a Line upon B and thro' the Point C, draw a Line Parallel thereto, produce AD untill it meets the Parallel CE; which being done, I say that DE is the third Proportional sought. For AB* is to AD, or to its equal BC, as AC is to AE; and therefore as † AC—AB is to AE—AD, that is, as BC is to DE; therefore $\div \div$ A B. B C. D E; which was required.

PROB. 5. THREE Lines being in Proportion, to find a fourth Proportional. Eucl. 6. Prop. 12.

THE First is AB, the Second AD, with which make any Angle at pleasure, such as BAD; the Third is BC which join to AB, so that they both together may make a right Line; then thro' B and D draw a right Line; and thro' the Point C draw the Line CE parallel thereto; prolong AD untill it meet the Parallel CE; which being done, I say that DE is the Fourth Proportional: for † AB. AC::AD. AE; therefore *dividedly*, § AB. AC—AB::AD. AE—AD; that is, that AB. BC::AD. DE, and consequently Alternately AB. AD::BC. DE.

PROB. 6. To find all the Lines that can be reciprocally Proportional to two given Lines. 25

LET the two given Lines be AB and AC; the shortest AB, must be taken from the Longest AC, and a Circle be described having the longest for its Diameter; then from the Point B, the End of the shorter Line, draw upon the Diameter an indeterminate Perpendicular, as Y. Every Line drawn from A which cuts the Indeterminate Line, and ends at the Circle as AD, or which cuts the Circle and ends at the Indeterminate Line, will solve the Problem, AE and AD are reciprocal to AB, AC as are also AF and AG; for BE is Antiparallel to DC, since AEB=ACD, and ABE

* Sup. n. 16. † B. 3. n. 48. ‡ Sup. n. 16.
§ B. 3. n. 48.

$ABE=ADC$: $\parallel$ therefore the Lines are cut reciprocally. * AF
 $C=ABG$, † and $ACF=AGE$. ‡ then BG and FC are Anti-
 parallels; therefore AC and AG are reciprocally divided.

26 PROB. 7. To describe upon a given Line a Figure Similar to a
 Figure given. Eucl. 6. Prop. 18.

THERE is no more required than to reduce the given Figure
 into Triangles § and to draw Lines upon the given Line making
 the same Angles, * and forming Similar Triangles.

27 LEMMA 1. If from the right Angle of a Rectangled Triangle
 be drawn a Line Perpendicular upon the Base, it will divide it in-
 to two other Triangles Similar to the whole Triangle, and also one
 to another. Eucl. 6. Prop. 8.

LET ABD be a Rectangled Triangle. If from the right An-
 gle A be drawn AC Perpendicular upon the Base BD , I say that
 the three Triangles ABD , ABC , ACD are all Similar to each
 other, 1. They are all right Angled. 2. The Angle B is com-
 mon to both the Triangles ABD and ABC ; therefore all their
 Angles are equal, † and consequently Similar. ‡ The Angle D ,
 is also common to both the Triangles ABD and ADC ; they are
 then also Similar, since ABC and ADC are Similar to a Third,
 they are Similar one to another; § consequently the three rect-
 angled Triangles ABD , BCA , CAD are Similar.

28 THEOR. 10. If a Line be drawn from the right Angle of a
 Rectangled Triangle, Perpendicularly upon the Hypotenuse, it will
 be as follows. (viz.) 1. The Perpendicular is a mean Proportional,
 between the two Segments of the Hypotenuse, 2. The longest Side of
 the Triangle is a mean Proportional between the Hypotenuse, and
 the greatest Segment. 3. The shortest Side of the Triangle is a mean
 Proportional between the Hypotenuse, and the lesser Segment.

By the preceeding Lemma, the Rectangled Triangle ABD ,
 is divided by the Perpendicular AC into two rectangled Trian-
 gles which are Similar thereto and also to one another; so that
 ABD , CBA , CAD are three similar Triangles, therefore 1. †
 $BC:AC::AC:CD$, or $\therefore BC.AC=CD$. 2. $CD:AD::AD$.
 BD , or $\therefore CD.AD=BD$. 3. $BC:AB::AB:BD$, or $\therefore BC$.
 $AB=BD$. Q. E. D.

29 PROB. 8. To find a mean Proportional, between two given
 Lines. Eucl. 6. Prop. 13.

METHOD 1. THE two given Lines are AB and BC ; join
 them so that they may make a right Line, from the middle
 thereof which is G , and with the Extent of AG or GC , de-
 scribe a Semi-circle, then upon B erect a Perpendicular, and pro-
 duce it till it reaches the Circumference to wit at the Point D ,
 I say, that BD is a mean Proportional between AB and BC .

The

$\parallel$ B. 2. n. 52. and 39. * Sup. n. 18. † B. 2. n. 52. and 39..

‡ B. 2. n. 39. and 53. § B. 2. n. 122.

* B. 2. n. 29. † B. 2. n. 80. ‡ Sup. n. 8.

§ Sup. n. 11. † Sup. n. 10.

The Angle ADC in the Semi circumference is a right Angle, * BD is Perpendicular by construction; then by the preceding Theorem BD is a mean Proportional between AB and BC; or $\therefore AB. BD. BC.$

METHOD 2. AB and CB are two given Lines; prolong AB so that $BD=AC$; and from D and A, as Centers, with the Extent of AB or CD; describe two Circles which intersect in E, the Line EB or EC is the mean Proportional between AB and CB. By construction $AB=AE$, and $CE=BE$: then EAB and CEB are both Isosceles; and the Angle ABE is common to each of them, these two Isosceles are then Equiangular, † and therefore Similar; ‡ then $AB. BE::BE. BC$, or $\therefore AB. BE. BC.$ §

PROB. 9. HAVING the three first Lines of a Geometrical Progression of Lines, to find others, ad inf. 30

LET the three progression Lines be, the First AC, the Second AD, the third AB, divide AB into two equal Parts; and upon the middle thereof with the Extent of half AB, describe a Semi-circle, upon the Line AB take a Part equal to the first Line AC; and upon C erect a Perpendicular which is bounded by the Circumference at the Point D, from whence draw a Line to the Point B, and an indefinite Line X from A thro' D, also prolong the Line AB ad inf. which I call Z; then at the Point B erect a Perpendicular which cuts X at the Point E, from whence draw a Perpendicular which cuts Z at the Point F, from which erect a Perpendicular which cuts X at the Point G, from whence let fall a Perpendicular GH, and so on for the Rest. 1. The Line AD being a mean Proportional between AC and AB, || is equal to the second Line given, which is hereby found, if it had not been given. 2. All the Triangles ADB, ABE, AEF, AFG, &c. are Equiangular, * since that beside their common Angle XAZ, they are all right Angled; for the Angle ADB in the Semi-circle is a right Angle, † and EF and HG are Perpendicular upon X by construction, as are DC. BE. GF upon Z; hence according to the Definition, ‡ they are Similar, and their Homologous Sides Proportional: § then $AC. AD::AD. AB$, and $AD. AB::AB. AE$, and $AB. AE::AE. AF$, and $AF. AG::AG. AH$, &c. consequently $\therefore AC. AD. AB. AE. AF. AG. AH$, &c.

PLATO'S Method of finding two mean Proportionals between two given Lines, mechanically.

THE given Lines are AB and AC, which join so as to make a right Angle. Place the Square X so that its Angular Point may be upon the produced Part of AB, and that one of its Sides be close to C the End of AC. Z is another Square which must

be

* B. 2. n. 44. † B. 2. n. 85. ‡ Sup. n. 9.

§ Sup. n. 10. || Sup. n. 28.

* B. 2. n. 80. † B. 2. n. 44. ‡ Sup. n. 9.

§ Sup. n. 10.

be disposed so that one of its Sides rub against X, and the other against the Point B the Extremity of AB: Hence the Triangles CDE and DEB, are right Angled, DA and EA are Perpendiculars, hence * $\therefore AC. AD. AE$, and $\therefore AD. AE. AB$; then $\therefore AC. AD. AE. AB$.

DES CARTES his Method for finding between two given Lines, as many Proportionals as you please.

THE Instrument he used is composed of several Squares, which are so adjusted one to the other, that when the Angle FAE is shut, or that the two Rules FA and AE touch one another, all the other Rules BC. CD. DF. EF. touch each other and slide to the Point A; if the Angle EAF be opened, these same Rulers are impelled and driven back, therefore two Lines being given, push the Ruler BC, so that AB may be equal to the shortest Line, and open the Angle EAF so that the Ruler DE may be distant from A the Length of the second Line. It is evident that AC and AD are two mean Proportionals between AB and AE. In order to find several mean Proportionals, the Number of Squares must be increased.

32 THEOR. 11. THE Segments of two Chords which intersect within a Circle, are in reciprocal Proportion.

LET the two Lines be BD, CE which intersect at the Point A. It must be proved that BA. CA::AE. AD. Let there be drawn BC, ED; which form the two similar Triangles BAC, EAD; for the Angles at A are equal, and the Angle B is equal to the Angle E, seeing they are placed upon the same Arc CD; therefore their Homologous Sides are proportional. † Then BA. CA::AE. AD; then Alternately, AB is reciprocally to AE, as CA to AD; which was to be proved.

33 LEMMA. 2. If a Tangent and a Secant be drawn from a Point out of the Circle, the Tangent is a mean Proportional between the whole Secant, and the Part thereof which is out of the Circle.

LET the Tangent be CB and the Secant CA; it must be proved that AC. CB::CB. CD, or $\therefore AC. CB. CD$. The Angle C is common to both the Triangles ACB and DCB, the Angle DBC is measured by half the Arc BD, ‡ the same half is also the Measure of the Angle BAC; § hence the two Triangles ACB and BDC having two Angles equal, and consequently their Third, they are Similar, and Proportional; ¶ the Side DC of the Triangle BCD is Homologous to the Side BC of the Triangle ACB; therefore AC. BC::BC. CD, or $\therefore AC. BC. CD$.
Q. E. D.

34 PROB. 10. To divide a Line, so that its longest part may be a mean Proportional between its shortest part and the whole Line, which is called mean and extreme Ratio. Eucl. 6. Prop. 30.

THE given Line is BC, at the Point B erect a Perpendicular BG, equal to half BC, with the Extent of GB, describe a Circle whose Diameter is consequently equal to BC, draw the Secant

* Sup. n. 28.

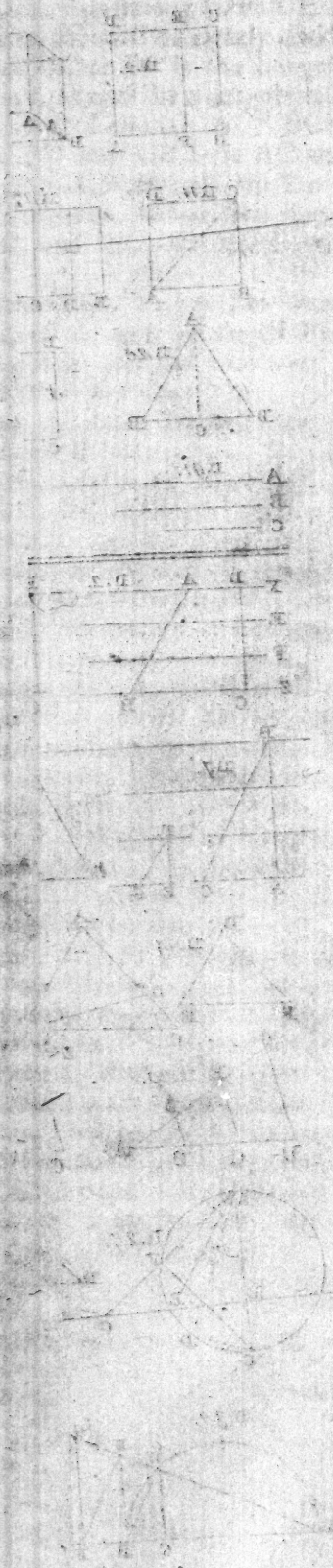
† Sup. n. 10.

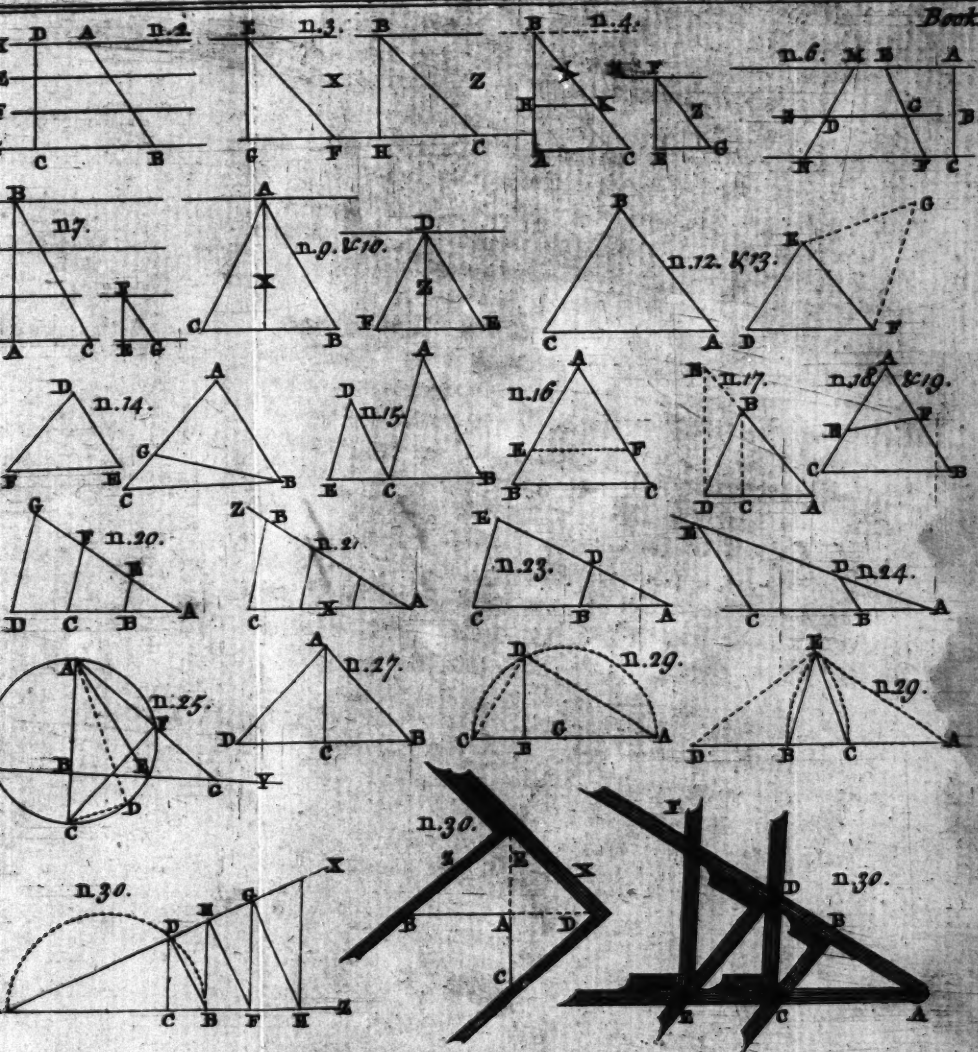
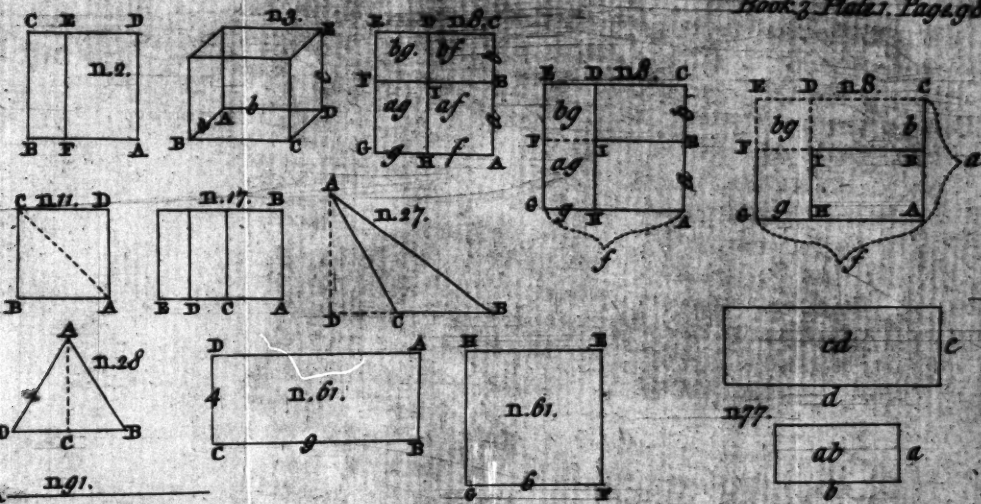
‡ B. 2. n. 37.

§ B. 2. n. 39.

¶ Sup. n. 8 and 10.

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cant AC, then having taken CH upon BC, equal to CD, I say that the Line BC will be rightly divided at the Point H, as is required. If HC or DC is the Longest part, and BH or BC—HC the Shortest, It must be demonstrated that $\therefore$ BC. CH. HB. by the preceeding Lemma, AC. BC::BC. CD, and subtracting the Lines BC and CD from AC and BC, the Remainders AC—BC and BC—CD are also in the same Proportion; * hence BC. CD::AC—BC. BC—CD. But by construction AC—BC=DC=HC, and BC—HC=HB; therefore $\therefore$ BC. CH. HB; Q. E. D.

COROLLARY 1. A Line being divided into mean and extreme Ratio, if the mean or longest Part be added thereto, this new Line will also be divided into mean and extreme Ratio, whose mean is the said first Line.

LET AC be a Line divided into mean and extreme Ratio at the Point B, the mean or longest Part is BC, which add to the Line so as to make CD equal to BC, It must be proved that $\therefore$ AD. AC. CD. Let AC=a, and CB=x; then a—x=A B, and CD=x, and a+x=AD. It must then be proved that $\therefore$ x. a+x. a. x. by the Hypothesis a. x::x. a—x. Permutatively x. a::a—x. x. Compoundedly, x+a. a::a—x+x. x. but a—x+x=Cypber, therefore x+a. a::a. x, or $\therefore$ x+a. a. x; which was to be proved.

COROLLARY 2. A Line being divided into mean and extreme Ratio, if its shortest part be Subtracted from the Mean, the Remainder will be also divided into mean and extreme Ratio.

THE Line AC is divided into mean and extreme Ratio, at the Point B, its shortest part is AB, it must be proved, that if from BC be subtracted DC=AB, the Remainder AD will be divided according to the same Ratio; that is, that $\therefore$ AD. AB. BD. Let AB=y, BC=x, BD=x—y. then AC=y+x, it must therefore be demonstrated that $\therefore$ x—y. y. x. According to the Hypothesis, y. x::x. y+x. by Permutation x. y::y+x. x. dividedly, x—y. y::y+x—x. x; as +x—x=0: Then x—y. y::y. x. therefore $\therefore$ x—y. y. x; which was to be proved.

COROLLARY 3. FROM the first Corollary it is easy to conclude that when a Line is divided into mean and extreme Ratio, innumerable other longer Lines may be divided into mean and extreme Ratio, by adding the mean to the whole first Line, and by the Second Corollary, an infinite number of shorter Lines may be divided into mean and extreme Ratio, by subtracting the shortest Part from the mean.

S E C T. III.

Of the Ratio's and Proportions which the Circumferences of two or more Figures bear to each other, and to the Radii of their circumscribing Circles.

DEFINITION.

38 **T**WO rectilinear Figures are called Similar, when their Angles are equal each to each, and the Sides containing them Proportional.

39 **COROLLARY.** HENCE it follows that all regular Figures of the same Name are Similar; for being of the same Name they have the same Number of Sides and Angles; and because they are regular, all the Angles are equal as well as the Sides containing them, * and therefore in an equal Ratio Proportional, therefore they are Similar, according to the Definition.

40 **THEOR. 1.** THE Circumferences of two similar Figures are one to another in the same Ratio, as their Sides, each to each.

LET there be two Hexagons X and Z, each Side of X is a ; hence its whole Circumference is $6a$; each Side of Z is b , and consequently its whole Circumference is $6b$; now $6a : 6b :: a : b$. †

41 **THEOR. 2.** THE Circumferences of two regular and similar Figures, are to each other as the Radii or Diameters of their circumscribing Circles, and as their Apotomes.

X and Z are two regular and similar Polygons; from the Centers of their circumscribing Circles, draw the Radii AB and AC, DE and DF; the two Triangles BAC and DEF are Isocles by construction; and since that these two Figures are Similar, the Angles at their Centers BAC and EDF are the same; therefore the Triangles are Similar, then AB, or the Radius of X is to DE, the Radius of Z, as BC to EF. Now the Circumference of X is to that of Z by the preecedent Theorem, as BC is to EF, therefore the Circumference of X is to that of Z, as the Radius of X is to that of Z, or as the Diameter of one is to the Diameter of the other; for the Radii and Diameters have the same Ratio one to another, the Radii being each the half of the Diameters. If AM is Perpendicular upon BC, it is the Apotome of X, ‡ so likewise if DN is Perpendicular from D upon EF, it is the Apotome of Z; now according to what hath been proved,

* B. 2. n. 106.

† B. 3. n. 54.

‡ B. 2. n. 144.

ed, * BC is to EF as the Apotome of X is to Z; therefore the Circumference of X is to that of Z, as the Apotome of X is to the Apotome of Z.

COROLLARY, *THE Circumferences of two Circles are one to another, as the Diameters or Radii thereof.*

CIRCLES may be esteemed as regular Polygons; now the Circumferences of two Polygons are one to another as their Diameters, therefore the Circumferences of Circles are one to another as their Diameters, and consequently as the Radii thereof, seeing the Radii are the Halves of the Diameters.

THEOR. 3. *THE Arcs of an equal number of Degrees taken in different Circles, are one to another as the Circles whose parts they are.* 43

THE Degrees are proportional Parts of the Whole; they are therefore one to another as the Wholes whose parts they are, that is, as the Circles of which they are the Degrees.

THEOR. 4. *THE Chords of similar Arcs, taken in different Circles, are one to another as the Arcs of which they are the Chords.* 44

RIGHT Lines being drawn from the Centers of the Circles to the Ends of the Chords, they will make similar Triangles, since the Chords of similar Arcs have their Angles at the Center equal; the Chords are then one to another as the Radii of the Circles; and therefore as the Circles, which are one to another as their Radii, † now by the preceeding Theorem, the Arcs of the same Number of Degrees are one to another, as the Circles of which they are the Parts, the Chords are therefore one to another, as the Arcs whose Chords they are.

THEOR. 5. *If from the Point of Contact where several Circles touch, there be drawn a right Line crossing the Circles, the Parts of that Line are one to another, as the Circles which it intersects.* 45

THE Point where the several Circles touch is A; from A let there be drawn a Line crossing the Circles C. D. E. It must be proved that the Parts of that Line AC. AD. AE. are one to another as the Circles which it intersects. Thro' A, draw the Tangent AB. hence the Angle BAE is measured by either the Arc CA, or AD, or AE, † therefore these three Arcs are Similar; the Chords AC, AD, AE, of similar Arcs, are by the preceeding Theorem, one to another as the Arcs AC. AD. AE, and these Arcs are one to another as their corresponding Circles; the Parts of the said Line are therefore one to another, as the Circles which it crosseth.

THEOR. 6. *ANGLES which are in the same or equal Circles, whether their Vertices be in the Circumference, or in the Center, are one to another as the Arcs upon which they are placed.* Eucl. 6. Prop. 33. 46

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* Sup. n. 10.

† Sup. n. 42.

‡ B. 2. n. 87:

This is very evident, by reason those Arcs are the Measures thereof. *

- 47 LEMMA. ONE of the acute Angles of a right angled Triangle being divided into as many equal parts you please, and right Lines drawn upon the opposite Side, those parts of that Side which are farthest from the Perpendicular are longest.

LET ABE be a right angled Triangle, whose Angle BAE is divided into as many equal parts you please, by right Lines drawn from the Point A to BE. It must be proved that the parts of B E which are farthest from AB, are longest; that the Part DC is longer than CB, and ED longer than DC. Seeing, that the Angle BAC was by the Hypothesis, made equal to the Angle CAD, the Angle BAD will be divided in two, and according to what has been demonstrated, † DC. CB::AD. AB. Now AD > AB; † then DC > CB. By the same Reason, if the Angle CAE is divided in two, as CAD is equal to DAE, then ED. DC::AE. AC; and since that AE > AC; then ED > DC; therefore the Parts of BE which are farthest from AB are longest.

- 48 THEOR. 7. OF two regular Polygons inscribed in the same or different Circles, that which hath most Sides, has an Apotome longer or shorter in Proportion to its Circumference.

LET the two Polygons be Z and Y, I suppose Y to have twice as many Sides as Z; that therefore if BC is a Chord of 20 Degrees, GH will be a Chord of 10, that is, if Z hath 18 Sides, Y hath 36. The Apotome of Z is AD, § and the Apotome of Y is FL; the Angle BAC is supposed double that of GFH; therefore DAC is equal to GFH, having divided the Angle DAC equally in two, the Angle DAE will be equal to LFH. The two Triangles LFH and DAE are right Angled, and have the Angles LFH, EAD equal; they are therefore Equiangular, || and Similar; * hence LH. LF::DE. DA. † the Polygon Y has 36 Sides; hence 72 times LH is its circumference, multiply DE by the same Number. If DE were the fourth part of BC, then 72 times DE would be the exact Circumference of the Polygon Z, which hath 18 Sides, and contains 18 times BC; but by the preceeding Lemma, DE is shorter than EC, therefore less than the fourth Part of BC; then 72 DE is less than the Circumference of Z, which I call *m*, wherefore 72 DE hath lesser Proportion to DA, but not *m*, which is longer, † but as has been demonstrated, LH. LF::DE. DA, also § 72 LH.LF::72 DE. DA; therefore 72 LH hath a less Proportion to the Apotome LF, than the said *m*, the Circumference of the Polygon Z hath to its Apotome DA; which was to be proved.

The preceeding Demonstration well understood, may be generally applied to all other Polygons, not only in multiple Ratio

* B. 2. 10 and 39. † Sup. n. 17. † B. 3. n. 59.

§ B. 2. n. 144. || B. 2. n. 80.

* Sup. n. 8. † Sup. n. 10. † B. 3. n. 91.

§ B. 3. n. 54.

to of their Sides, but also in any Proportion whatsoever, as 5 to 13, or what other you will, which cannot be only of Number to Number, hence instead of (as in the proposed Example) the Angle DAC being double DAE, it will be here as 13 to 5, and it may be supposed as divided into 13 equal parts, of which the Angle DAE contains 5; and by the same Lemma, the five Portions of the Part DE which are nearest to the Perpendicular DA, are always in lesser Proportion, than the 13 of the Line DC; and therefore 26 times DE are always less, than ten times DC, which make the Circumferences of the Polygons, whose number of Sides is 13 and 5, DE and DC being each only half their Sides, and therefore the Conclusion will always be the same as before.

COROLLARY 1. *Of two Polygons having the same Circumference, that which has the most Sides hath the longest Apotome.* 49

COROLLARY 2. *The Radius of a Circle having the same Circumference as a Polygon, is longer than the Apotome of that Polygon.* 50

A Circle is a Polygon having infinite Sides, whose Apotome is its Radius, which is therefore longer than the Apotome of any other Polygon, according to the preceding Proposition.

SECT. IV.

Of the Ratio's and Proportions of Surfaces.

THEOREM 1.

IN every Parallelogram, the adjacent Parallelograms, or those which are about the Diameter, are similar to each other, and also to the whole Parallelogram. Eucl. 6. Prop. 24.

LET ABCD be a Parallelogram, whose Diameter is AC, about which there be two other Parallelograms AGFK and CEFK. 1. Each of them hath one Angle common with ABCD. The opposite Angles of the Parallelograms are equal; * therefore $\angle GAK = \angle ECH = \angle HFE$, and $\angle KAG = \angle KFG$: Hence the three Parallelograms having two of their Angles equal, the others are also equal; for their Sides being Parallel, they make the same Angles. † The Triangles ABC, AGF, FEC, are therefore Equiangular, consequently Similar, ‡ as also ADC, AKF, EHC; then § $AG : GF :: AB : BC$, and $AG : AF :: AB : AC$, and $AF :$

AK

* B. 2. n. 119. † B. 2. n. 27. ‡ Sup. n. 8.

§ Sup. n. 10.

AK::AC. AD. so likewise AK. AG::AD. AB; therefore AK FG and ABCD are Similar; so likewise CEFH, and ABCD are Similar. *

- 52 THEOR. 2. IF from a Parallelogram be taken away a Similar Parallelogram, and alike posited in regard to the whole, and having one Angle common thereto, the Parallelogram taken away is about the Diameter of the whole Parallelogram. Eucl. 6. Prop. 26.

LET the Parallelogram AGFE be subtracted from the Parallelogram ABCD, they are Similar, and have the Angle EAG common. It must be proved that their Diameters are in the same Line. If it be denied, and it be asserted that the Diameter AHC intersects EF at the Point H; draw HI, parallel to AE, the Parallelograms EI and DB are Similar, † therefore A E. EH::AD. DC::AE. EF. ‡ then EH=EF. § Hence the Part is equal to its whole; which is absurd.

- 53 THEOR. 3. WHEN a Line is divided into mean and extreme Ratio, the Rectangle of the whole Line and its lesser part is equal to the Square of the mean Part.

LET AB be a Line divided into mean and extreme Ratio, the mean Part is AD. then AB. AD::AD. DB, or $\div \div$ AB. AD.

$\frac{-2}{BD.}$ therefore $AD=AB \times DB$; || which was to be proved.

- 54 COROLLARY. To divide a right Line, so that the Rectangle of the whole Line and one of its parts, may be equal to the Square of the other part. Eucl. 2. Prop. 11.

THIS is only to cut the given Line into mean and extreme Ratio, according to the Theorem.

- 55 THEOR. 4. IF two right Lines intersect within a Circle, the Rectangle comprehended under the two Parts of the one, is equal to the Rectangle contained under the two Parts of the other. Eucl. 3. Prop. 35.

THE two Chords BD and CE of the Circle X intersect at the Point A, it must be proved that $BAXDA=CAXEA$. It has been proved * that BA. AE::AC. AD. then BAXAD, the product of the Extremes, =AEXAC, the Product of the of Means. † Q. E. D.

- 56 THEOR. 5. IF from a Point any where taken out of a Circle, be drawn two right Lines, one touching the Circle and the other cutting it, and bounded by the concave Circumference, the Rectangle comprehended under the whole cutting Line and its part that is out of the Circle, is equal to the Square of the Tangent, or touching Line. Eucl. 3. Prop. 36.

FROM the Point C which is out of the Circle, let there be drawn two right Lines, CB which touches the Circle, and CA

* Sup. n. 38. † Sup. n. 51. ‡ Sup. n. 38.

§ B. 3. n. 52. || B. 3. n. 57.

* Sup. n. 32. † B. 1. n. 56.

CA which cuts it. It must be proved that $AC \times CE = BC^2$, it has been demonstrated, * that $AC : BC :: DC : BC$: therefore AC

$\times CD = BC^2$; † which was to be proved.

COROLLARY 1. If from a Point taken at pleasure out of a Circle, be drawn ever so many right Lines, which cross the Circle and end at the concave Circumference; the Rectangle comprehended under any one of the cutting Lines, and its part out of the Circle, is equal to the Rectangle contained under any other cutting Line, and its part out of the Circle.

For each of the Rectangles is equal to the Square of the cutting Line, which is drawn from the same Point.

COROLLARY 2. If from a Point taken at pleasure out of a Circle, be drawn two Tangents, or two right Lines which touch the Circle, they are equal to each other.

For the Square of each of the Tangents is equal to the Rectangle of a Secant Line and its part out of the Circle; and therefore these Squares are equal to each other; from hence it follows, that the Lines which are the Sides thereof, are equal.

THEOR. 6. If from a Point taken at pleasure out of a Circle, be drawn two right Lines, one of which cuts the Circle and ends at the concave Circumference, and the other only reaches to the Circle; and that the Rectangle comprehended under the whole Secant or cutting Line and its part out of the Circle, be equal to the Square of that Line which only reached the Circle, then this last Line will be a Tangent to the said Circle. Eucl. 3. Prop. 37. same fig.

SEEING the Square of the Line which just reached the Circle, is equal to the Rectangle $AC \times DC$, it is equal to the Tangent BC , whose Square is equal to that Rectangle, it cannot then be a different Line, therefore it is a Tangent: which was to be proved.

THEOREM 7. If from a Point in the Circumference of a Circle two Lines be drawn to its concave Circumference, and there be a third Line drawn cross those two Lines, to two Points in the Circumference, equally distant from the given Point, the Rectangle comprehended under one of the Lines and its part contained between the given Point and third Line, is equal to the Rectangle of the other Line, and its part contained between the said given Point and third Line.

LET K be the Point in the Circumference, from which let there be drawn the Lines KF and KG, and thro' the Points E and D which are equally distant from K, draw the Line ED, it must be proved that $KF \times KB = KG \times KC$. The two Lines BC and FG are Antiparallels; for the Angle KBC is measured by half the Arc EF, more half the Arc KD, or its equal KE; † now the half of KF is also the Measure of KFG: § therefore

P

KBC

* Sup. n. 33.

† B. 3. n. 57.

‡ B. 2. n. 52.

§ B. 2. n. 39.

$KBC = KGF$. by the same reasoning $KCB = KFG$. Hence FG and BC are Antiparallels, according to the Definition thereof; * then $KF : KC :: KG : KB$; † therefore $KF \times KB = KG \times KC$. ‡ which was to be proved.

61 COROLLARY. THE same Things being supposed as above, it not only follows that if from the Point K there be drawn Lines *ad. inf.* as KF or KG , which terminate at the concave Circumference, and are crossed by the right Line ED , all the Rectangles made in the same Manner as above, are equal one to another, seeing these Lines taken two and two, are each equal to the Rectangle $FK \times KB$; but also, that if from the Point K be drawn the right Line KE , the Square of the said Line KE will be equal to each of the aforementioned Rectangles; for having drawn the right Line EF , it will always form two similar Triangles such as FKE , and EKB , whose Angle K is common, and the two other Angles also equal; viz. BFK to KEB , and FEK to EBK , § each standing upon equal Parts of the Circle's Circumference, hence their homologous Sides are Proportional; and || therefore $FK : KE :: KE : KB$; then the Rectangle $FK \times KB$ is equal to the Square of KE , † and it also follows, that if the Line ED is a Diameter, the Line KE will be the Side of a Square inscribed in the Circle, ‡ to which each of the Rectangles contained under the said Lines, such as $FK \times KB$ or others, are equal.

62 THEOR. 8. IF a Quadrilateral be inscribed in a Circle, the Rectangle contained under the Diagonals is equal to the Sum of the Rectangles made of the opposite Sides.

LET the Quadrilateral be $ABCD$, whose Diagonals are AC and BD ; it must be proved that $AC \times BD = BC \times AD + AB \times DC$. Draw BE , so that the Angle ABE may be equal to CBD , and that each of them being subtracted from the Angle ABC , the Remainders CBE and ABD may be equal; hence as the Angles ADB and ACB stand upon the same Arc, they are equal; § the Triangles BDA and BCE are Equiangular, || and Similar; * therefore $BD : AD :: BC : CE$; † hence the Rectangle of the Extremes is equal to that of the Means; and Consequently $BD \times CE = AD \times BC$. ‡ By the same Reason, the Triangles BDC and BAE are Similar; for $ABE = DBC$ by construction, and $BAC = BDC$ as standing upon the same Arc BC , their Homologous Sides are then also Proportional; therefore $BD : CD :: AB : AE$; hence $BD \times AE = CD \times AB$; § Now $BD \times AE + BD \times CE = BD \times AC$, || then seeing $BD \times AE = CD \times AB$, and

* Sup. n. 18. † Sup. n. 19. ‡ B. 3. n. 56.

§ B. 2. n. 43. || Sup. n. 10.

† B. 3. n. 56. ‡ B. 2. n. 124. § B. 2. n. 40.

|| B. 2. n. 80.

* Sup. n. 8. † Sup. n. 10. ‡ B. 3. n. 56.

§ B. 3. n. 56. || B. 3. n. 17.

and $BD \times CE = AD \times BC$, it must be therefore that $CD \times AB + AD \times BC = BD \times AC$, two things equal to a third, being equal one to another. Q. E. D.

THEOREM 9. If the Diameter of a Circle be divided, so that one of its parts be Quadruple the other, and there be raised upon the Point of division a Perpendicular bounded by the Circumference, I say, that the Square of the Chord drawn from the End of the Diameter to the End of the Perpendicular, is equal to five times the Square of the lesser Part of the Diameter. 63

LET NM be the Diameter of the Circle X so divided in A, that $NA = 4AM$, if the Perpendicular AB be erected and there be drawn the Chords NB, and BM, forming the right angled

Triangle NBM; * I say, that $MB = 5AM$. Let $AM = a$, then, according to the Supposition, $MN = 5a$, let $MB = b$, then $\div 5 a. b. a$, † therefore $bb = 5aa$; ‡ now let $BA = d$; hence $\div a. d. 4a.$ § by the same Reason therefore $4aa = dd$; which was to be proved.

COROLLARY 1. THE Diameter NM being divided into ever so many parts, the Square of MB will be equal to as many times that of each of the Parts, as there are parts. 64

IF, as above and same Figure. NM is so divided at the Point A, that AM be the Fifteenth (or any other part) of NM, it will always be, $\parallel \div MA. MB. MN$, or $\div 5 a. MB. a$; therefore the Square of MB is also equal to $5aa$. *

COROLLARY 2. THE Square of BA is equal to so many times the Square of each of the Parts, as there are parts less one. 65

FOR *fig. proceed.* Supposing always the same, or any other Division whatsoever, the Proportion will continue to be the Same, $\div AM. BA. AN$, or $\div a. BA. 4a.$ † then $4aa$ or $5aa - aa$ is equal to the Square of AB, ‡ that the Square of AB is equal to so many times the Square of each of the Parts, as there are parts less one, is what was to be proved.

THEOR. 10. IF a Line be divided into mean and extreme Ratio; I say that the Square of half the Line and its greatest Segment, is equal to five times the Square of the said half Line. Encl. 13. Prop. 1. 66

LET BA be divided into mean and extreme Ratio at the Point C, and let $AB = 2a$, and $BC = b$, then $CA = 2a - b$, it must be demonstrated that the Square of $a + b$ (which is $aa + 2ab + bb$) is equal to $5aa$, taking away aa from each part, there will remain only to prove that $2ab + bb$ is equal to $4aa$. By the Supposition $2a. b :: b. 2a - b$; § then $bb = 4aa - 2ab$, || and adding $2ab$ to each part, it will be $bb + 2ab = 4aa$; * which remained

to

* B. 2. n. 44. † Sup. n. 28. ‡ B. 3. n. 57.

§ Sup. n. 28. || Sup. n. 28.

* B. 3. n. 57. † Sup. n. 28. ‡ B. 3. n. 17.

§ Sup. n. 34. || B. 3. n. 57.

* B. 3. n. 6.

to be proved; then substituting aa which was before taken away, it will be $aa + bb + 2ab = 5aa$; § E. D.

67 THEOREM 11. Two right Lines being divided into mean and extreme Ratio, they are similarly divided. Eucl. 14. Prop. 2.

LET the two Lines be Z and X divided into mean and extreme Ratio at the Points C and G . Hence $Z. AC. AC. CB$, and $X. EG. EG. GF$. Let $AC = a$, and $CB = c$, and $EG = m$, and $GF = n$, it must be demonstrated that $AC. CB :: BG. GF$, or that $a. c :: m. n$. As the Square of half Z together with a , is equal to five times the Square of half Z ; so likewise the Square of half X together with m , is equal to five times that of half X , hence these Squares being Proportional, their Sides are also Proportional * seeing they are in Subduple Ratio, wherefore half $Z + a$. half $Z ::$ half $X + m$. half X : and dividedly $+ a$. half $Z :: m$. half X , the Consequents being changed and doubled † $Z. a :: X. m$, but by the Supposition $÷ Z. a. c$ and $÷ X. m. n$, therefore the two Lines are divided Proportionally; § E. D.

68 THEOR. 12. A Line being divided into two unequal parts, if the Quintuple of the Square of the half thereof is equal to the Square of that half together with its longer part, the Line is then divided into mean and extreme Ratio, whose mean part is the greater Segment Eucl. 13. Prop. 2.

LET AB be a Line unequally divided into two parts at the Point C , let $AB = 2a$, and $AC = b$, hence $CB = 2a - b$. It must be demonstrated that if five times the Square of a , the half of AB , is equal to the Square of $a + b$, that is, to the Square of half a together with b , which I express thus $5aa = aa + 2ab + bb$, then the Line AB is divided into mean and extreme Ratio, and AC or b , will be its longer part, to prove this it must be shewn that if AB is so divided, then $5aa = aa + 2ab + bb$. Let us suppose then that $÷ 2a. b. 2a - b$; then $4aa - 2ab = bb$; and by adding $+ 2ab$ to each part, it will be $4aa = 2ab + bb$, || also by adding aa to each part, it will be $5aa = aa + 2ab + bb$, which was to be proved.

69 THEOR. 13. If a right Line is divided into mean and extreme Ratio, the Square of the lesser Part together with half the greatest, is Quintuple the Square of half the greater part. Eucl. 13. Prop. 3.

LET AB be a Line cut into mean and extreme Ratio at the Point C , (fig. preceed.) Let $AC = 2m$, and $CB = n$, then $AB = 2m + n$. It must be demonstrated that the Square of half the greatest Part AC , together with the lesser part BC , that is, the Square of $m + n$, (which is $mm + 2mn + nn$) is Quintuple the Square of half AC , that is, of m , which is mm , according to the Supposition $÷ 2m + n. 2m. n$; then $2mn + nn = 4mm$; * adding

* B. 3. n. 84. + B. 3. n. 51. + B. 3. n. 45.

§ B. 3. n. 53. || B. 3. n. 55.

* B. 3. n. 57.

mm , to each part it will be $mm+2mn+nn=5mn$; Q. E. D.

THEOR. 14. If a right Line is divided into mean and extreme Ratio, the Square of the whole Line, together with that of its lesser part, is triple the Square of the greater part. Eucl. 15. Prop. 4.

LET AB be divided into mean and extreme Ratio, at the Point C, (fig. preced.) Let $AB=Z$, and $AC=a$, and $BC=e$; then $\therefore Z. a. e.$ It must be demonstrated that $ZZ+ee=3aa$. 1. $ZZ=aa+2ae+ee$. * It must therefore be demonstrated that $aa+2ae+ee=3aa$. 2. $ae+ee=Ze$, † and $Ze=aa$, ‡ by the Supposition, hence $ae+ee=aa$, there may then in order to obtain a Proof, be substituted in the Equation in the Room of $2ae+2ee$, their equal value $2aa$, and it will be $aa+2aa=3aa$; which was to be proved.

THEOR. 15. SIMILAR Triangles and Parallelograms, are one to another in Ratio duple that of their Sides, which are of the same Ratio. Eucl. 6. Prop. 19.

LET ABD and EFG be two similar Triangles, let fall the Perpendiculars AC and GH; the Triangles EGH, and BAC, being right Angled, and the Angle B being equal to the Angle E, they are Similar; then $BC. AC::EH. GH$. § also the two Triangles DAC and FGH being Similar; then likewise $CD. AC::FH. GH$, hence $BC+CD. AC::EH+HF. GH$. || Let $AC=b$, and $BD=d$, $GH=m$, and $EF=n$, hence $b. d::m. n$; and Alternately, $b. m::d. n$. * The Surface of ABD is the half of bd , † and that of GEF is the half of mn ; but the Ratio of bd to mn , is compounded of the two Ratio's of b to m , and of d to n ; ‡ now these two Ratio's are the same: Therefore, according to the Definition of a duple Ratio, § the Ratio of bd to mn , is duple. Let BDMN and EFKI be two similar Parallelograms, by the same Reasoning it will be proved that $b. d::m. n$, and $b. m::d. n$, the Surface of BDMN is bd , and that of EFKI is mn ; the Ratio of bd to mn , is composed of the two Ratio's of b to m , and of d to n ; these two Ratio's are equal; the Ratio compounded thereof is therefore duple.

LEMMA. SIMILAR Polygons may be divided into an equal number of Triangles, similar each to each.

LET the two similar Polygons be ABCDE, and FGHIK; I say that they may be divided into an equal number of Triangles, each Similar to its corresponding one; for 1. Seeing the Polygons are Similar, they have each an equal number of Sides, and the Angles which they contain are equal by the Definition; || hence from one of the equal Angles, as from E and K, may be drawn Lines to the other Angles which will divide the Polygons into an equal

* B. 3. n. 20. † B. 3. n. 19. ‡ B. 3. n. 57.

§ Sup. n. 10. || B. 3. n. 63.

* B. 3. n. 46. † B. 2. n. 135. ‡ B. 3. n. 77.

§ B. 3. n. 72. || Sup. n. 38.

equal number of Triangles. * 2. All the Triangles of one Polygon are Similar to those of the other, each to its corresponding one; thus the Triangle ABE is Similar to its corresponding one FGK; for by the Supposition the Angle EAB = KFG; and their containing Sides are Proportional; that is, EA. AB::KF. FG; and therefore the two Triangles are Similar. † It is the same of the other Triangles, whose Angles contained between homologous Sides are equal, whether they be form'd by the Sides of the Polygons, or by the remaining ones after the equal ones which touch at the Points E and K are taken away, such as BEC and GKH, therefore &c. which was to be proved.

73 THEOREM 16. ALL similar rectilinear Figures, are one to another as the Squares of their homologous Sides.

LET the same things and the same Figure be supposed as in the preceding Lemma. 1. Each of the Figures may be divided into an equal number of Triangles similar one to another, each to each, by the preceding Lemma, &c. 2. Each of the Triangles of one Figure will be to each of its corresponding ones in the other Figure, in duple Ratio of each of the homologous Sides, by the preceding Theorem, † that is, in duple Ratio of AE to FK, or AB to FG, but the Square of AE is to that of FK, in duple Ratio of AE to FK; § and it is the same of all the other Triangles which compose the two Figures, whose Sides preserve the same Proportion by reason of their likeness, wherefore as each Triangle of the one is to its like in the other, so are all the Triangles of one to all the Triangles of the other, || that is, the first Figure is to the Second, as the Square of the Side AE is to that of FK; and it is the same of all the Rest pitch upon which you will, therefore. &c. which was to be concluded.

74 COROLLARY. If four Lines are Proportional, similar Figures described upon those Lines are Proportional, and if the similar Figures are Proportional, the four Lines are Proportional. Eucl. 6. Prop. 22.

LET the four proportional Lines be $a. b::c. d$, upon which let there be described four similar Figures V. X. Y. Z. by the preceding Theorem $V. X::aa. bb$, and $Y. Z::cc. dd$, but $aa. bb::cc. dd$; * therefore $V. X::Y. Z$, and if this be so, then $a. b::c. d$; for the equal duple Ratio's are compounded of equal Ratio's, therefore if $aa. bb::cc. dd$, it must be that $a. b::c. d$.

75 THEOREM 17. TRIANGLES and Parallelograms having the same Height, are one to another in the same Ratio as their Bases.

LET Z and X, be either two Parallelograms or two Triangles, having the same Height which I call a , the Base of Z is b , and that of X is d , the Surface of Z is ab , if it is a Parallelo-

gram;

* B. 2. n. 122. † Sup. n. 13. ‡ Sup. n. 71.

§ B. 3. n. 80. || B. 3. n. 50.

* B. 3. n. 84.

gram; but only the Half thereof, if it is a Triangle: * *ad* is also the Surface of X; or if it is a Triangle, then only the Half thereof; now *ab. ad::b. d*; † therefore Z and X are one to another, as their Bases.

PROBLEM 1. A Point being given in one of the Sides of a Triangle, to draw a Line thro' that Point so as to divide the Triangle according to a given Ratio. 76

LET ABC be a Triangle required to be divided according to a given Ratio, by drawing a right Line thro' the Point D. Let this Ratio be that of BG to GC; if D and G were one and the same Point, there should be drawn a Line from D to A, for BAD and DAC are one to another as BD and DC; † therefore what was required is done. If G and D are two different Points, draw GH parallel to AD, and from D to H, where that Parallel cuts AC, draw another Line *viz.* DH; it must be proved that it divides the Triangle ABC, according to the Ratio of BG to GC. The two Triangles BAG and GAC are one to another as BG to GC, § it is then required only to prove that DHC=AGC, and DBAH=BAG; hence as DHC. AGC::DBAH. BAG; the two Triangles GAH and GHD upon the same Base GH, and between the same Parallels are equal; || then GHC+CDH, or DHC=GHC+GAH, or GAC, it remains to prove that ABDH=ABG; which is easy; for by the same Reason as before mentioned, the Triangle ADG is equal to that of AHD; therefore adding to each the Triangle ABD, it will be ABDH=ABG; and therefore ABDH. DHC::ABG. ABC, therefore that is done which was to be done.

PROBLEM 2. To divide a Parallelogram according to a given Ratio, by drawing a Line thro' a given Point thereof. 77

LET the Parallelogram be ABCD, it is required to be divided according to the Ratio of BF to FC, by drawing a Line thro' the given Point E. 1. Draw FG Parallel to AB, or to CD, the two Parts ABFG and CDGF are one to another, as BF is to FC, according to the last Theorem, * 2. Take GH equal to FE, and having drawn a right Line from E to H, I say that ABEH and ECDH are the Parts required: For HGI and EFI are two similar Triangles; seeing they are Equiangular: For the Angles at the Point I are equal, † and the other Angles are likewise equal, being alternate ones between the same Parallels; ‡ therefore the Triangles having the Sides EF and GH equal, are entirely equal. § Therefore ABFIH=ABFG-HGI; and there being added to each the Value of HGI equal to FIE, it will be ABFIH+FIE=ABFG; which was to be demonstrated. It is proved the same Way that CDHE=CDGF.

Theorem

* B. 2. n. 133. † B. 3. n. 54. ‡ Sup. n. 75.

§ Sup. n. 75. || B. 2. n. 134.

* Sup. n. 75. † B. 2. n. 21. ‡ B. n. n. 25.

§ B. 2. n. 96.

78 THEOREM 18. THE Square of the Hypotenuse of a right angled Triangle is equal to the Sum of the Squares of the two other Sides. Eucl. 1. Prop. 47.

A Demonstration different from that laid down in Book 2. n. 142.

ABC is a right Angled Triangle, the right Angle is A, and

the Hypotenuse is BC. It must be proved that $\overline{BC} = \overline{AB} + \overline{AC}$; and that differently from what has been done already, * From the right Angle A, let fall the Perpendicular AD and prolong it, then according to what has been proved. † $\therefore$ BC. BA. BD; then ‡

$\overline{AB} = \overline{BC} \times \overline{BD}$, or $\overline{BE} \times \overline{BD}$ since that $\overline{BC} = \overline{BE}$, because it is a

Square: So likewise $\therefore$ BC. AC. CD; therefore $\overline{AC} = \overline{BC} \times$

CD, or $\overline{CF} \times \overline{CD}$ seeing that $\overline{CF} = \overline{CB}$; now $\overline{BC} = \overline{BE} \times \overline{BD} +$

$\overline{CF} \times \overline{CD}$: § Therefore $\overline{BC} = \overline{AB} + \overline{AC}$; which was to be 79 proved.

COROLLARY 1. WHAT Figure soever, is made upon the Hypotenuse of a right angled Triangle, it is equal to the similar Figures described and in a like Position upon the other two Sides thereof. Eucl. 6. Prop. 31.

UPON the Sides BD. DA. AB, let there be three similar Figures X. Y. Z, let X be upon the Hypotenuse, these Figures are one to another as $\overline{BD}$, $\overline{AD}$, $\overline{AB}$. || now $\overline{BD} = \overline{AD} + \overline{AB}$ by 80 the Theorem; therefore $X = Y + Z$.

COROLLARY 2. A Perpendicular being drawn from the right Angle of a rectangled Triangle upon the Hypotenuse, the Squares described upon the other two Sides are one to another as the Parts or Segments of the Hypotenuse.

LET ABD be a rectangled Triangle, whose Hypotenuse is DB, upon which the Perpendicular AC falls from the right Angle A. It must be proved that $\overline{AD} : \overline{AB} :: \overline{DC} : \overline{CB}$, since that $\therefore$ DB. DA. DC; * then $\overline{DB} \times \overline{DC} = \overline{DA}$; † by the same Reason $\therefore$ DB. BA. BC; then also $\overline{DB} \times \overline{CB} = \overline{BA}$, wherefore $\overline{DB} \times \overline{DC} : \overline{DB} \times \overline{BC} :: \overline{AD} : \overline{AB}$; but $\overline{DB} \times \overline{DC} : \overline{DB} \times \overline{BC} :: \overline{DC} :$

81 $\overline{CB}$; ‡ hence, $\overline{AD} : \overline{AB} :: \overline{DC} : \overline{CB}$, Q. E. D.

THEOREM 19. PARALLELOGRAMS having one Angle common or equal, are in a compound Ratio of the containing Sides. Eucl. 6. Prop. 23.

The

* B. 2. n. 142. † Sup. n. 28. ‡ B. 3. n. 57.

§ B. 3. n. 18. || Sup. n. 73.

* Sup. n. 28. † B. 3. n. 57. ‡ B. 3. n. 54.

THE Angle ABC is equal to the Angle GHI, I say that these two Parallelograms are in Ratio compounded of the Ratio's of AB to GH, and of BC to HI, the Parallelograms ABCE and GH IK are equal to the Rectangles BDFC and HLMI each to each, * which are in Ratio compounded of the Ratio's of BD to HL, and of DC to HI, † now the two Triangles ABD and GHL being Similar, the Ratio of DB to HL is the same as that of AB to GH, for. 1. They are Rectangled. 2. The Angles ABC and GHI being equal, if the right Angles DBC and LHI be taken away, the Remainders ABD and GHL are equal; hence they are Equiangular, ‡ and therefore their Sides are Proportional. §

COROLLARY 1. WHEN two equal Parallelograms have one Angle common or equal, the containing Sides thereof are in reciprocal Proportion. Bqcl. 6. Prop. 14.

IF the Parallelograms ABCE and GHIK are equal one to another, fig. above, I say that AB. BC. GH. HI, are in reciprocal Proportion; that is, that AB. GH::HI. BC. Seeing that $BD \times BC = HL \times HI$, then $\parallel BD. HL::HI. BC$, now since that the Angles ABC and GHI are equal by the Supposition, the Conclusion will be * that AB. GH::BD. HL::HI. BC; and therefore AB. GH::HI. BC; † 2. E. D.

COROLLARY 2. IF the Surfaces of two Triangles are equal, 83 and they have one Angle common or equal, The Sides containing that Angle are in reciprocal Proportion. Eucl. 6. Prop. 15. (same figure)

LET the two Triangles be ABC and GHI whose Surfaces are equal, and the Angles ABC and GAI equal, it must be demonstrated that AB. GH::HI. BC, these Triangles are half the Parallelograms ABCD and GHIK, which proves that AB. GH::HI. BC; which was to be proved.

PROB. 2. To find a Square equal to a given Rectangle. 84

LET bd be the given Rectangle suppose x to be a mean Proportional between b and d , † then $\div b. x. d$, and $bd = xx$, § thus is a Square found equal to a given Rectangle.

PROB. 4. To describe upon a given right Line, a right lined Figure similar to a given rectilinear Figure, and in a like Position thereto. Eucl. 6. Prop. 28. 85

LET AB be a right Line, upon which must be described a similar Figure and in a like Position to the Rectilinear one CDFE, reduce CDFE into two Triangles, $\parallel$ upon AB make the Triangle ABH, similar to the Triangle CDF, ¶ and upon AH the Triangle AGH similar to CEF, and so on, if the Figure had contained

* B. 2. n. 129. † B. 3. n. 77. ‡ B. 2. n. 80.

§ Sup. n. 10. $\parallel$ B. 3. n. 58.

* Sup. n. 7. † B. 3. n. 53. ‡ Sup. n. 89;

§ B. 3. n. 57. $\parallel$ B. 2. n. 122.

¶ B. 2. n. 97.

tained more Triangles. the Sides of these similar Triangles are Proportional, † but as they form or compose the two Figures, their Angles are therefore also equal, and their homologous Sides proportional, thus $AB : AG :: DC : CE$, and consequently the Figures are Similar; ‡ which was to be proved.

- 86 PROP. 5. Two rectilinear Figures being given, to describe a third Similar to one of them and equal to the other. Eucl. 6. Prop. 25.

THE given rectilinear Figure is A, 'tis required to make another Similar thereto and equal to B, another given right lined Figure. Upon CD describe the Parallelogram CE equal to A, § and likewise upon DE the Parallelogram DH equal to B having the given Angle D, then find IK a mean Proportional between CD and DG, || upon IK, describe L similar to A; ¶ I say that L is the Figure required, it must be proved that L is equal to B. By construction A and L are Similar to each o-

ther, therefore they are one to another as CD is to IK , ¶ that is, in duple Ratio of CD to IK ; † but by the Hypothesis $CD : IK :: DG$; hence CD is also to DG in duple Ratio of CD to IK . † and consequently $CD : DG :: A : L$, but also $CD : DG :: CE : DH$. † and by construction $CE = A$, and $DH = B$; then CD .

$$DG :: \begin{cases} A. B. \\ A. L. \end{cases} \text{ wherefore } B = L; \S \text{ 2. E. D.}$$

OBSERVE that this Construction would have been easier, if instead of the indifferent Parallelogram CE, there had been made a Rectangle.

- 87 THEOR. 20. A Square whose Circumference is equal to that of a right angled Parallelogram, is greater than the Parallelogram by the Square of half the Difference, which is between the Sides of the Parallelogram.

THE Square is X, and Z a Parallelogram having an equal Circumference, the Sum of the two Sides either of X or Z is AC, the half of AC, which is AB, is the Side of the Square X. The longest Side of Z is AE, and the Shortest EC. Let AB or $BC = b$, and $BE = a$, and that BD be taken equal to a , then AE the longest Side of Z will be $b + a$, and the shortest EC or AD will be $b - a$, their difference will then be $2a$, the Square X is therefore equal to bb , and the Rectangle Z equal to the Product of $b + a \times b - a$, that is, to $bb - ab + ab - aa$ or $bb - aa$; consequently the difference between X and Z is aa , the Square of half the difference of the Sides; which was to be proved.

Theorem

† Sup. n. 10. † Sup. n. 38. § B. 2. n. 139 and 140.

|| Sup. n. 29. ¶ Sup. n. 85.

¶ Sup. n. 73. † B. 3. n. 80. † B. 3. n. 72.

† Sup. n. 75. § B. 3. n. 52.

THEOR. 21. OF two regular Polygons having the same Circumference, that which hath most Sides hath the greatest Surface. ⁸⁸

A Polygon is equal to a Triangle whose Base is the Circuit of the Polygon, and its Perpendicular height the Apotome thereof. * Let there be therefore two Polygons of the same Circuit; the Half of which I call x , and let b be the Apotome of that which hath most Sides, and let that of the other be d ; then their Surfaces are xb and xd ; † Now $xb. xd :: b. d$, ‡ and $b > d$; § therefore xb is greater than xd .

CORALLARY. THE Circle is the most Capacious of all Hyperimetric Figures, that is, having the same Circumference. ⁸⁹

FOR a Square is greater than a Parallelogram of the same Circuit, || the Square is a Polygon, and consequently the Circle considered as a Polygon of an infinite number of Sides, has the greatest Apotome, therefore a greater Surface. It is the same of all other Polygons.

THEOR. 22. REGULAR and similar Polygons are in Ratio duple that of their Sides. Eucl. 6. Prop. 20. ⁹⁰

LET the two similar Polygons be X and Z ; they are equal to two similar Triangles, * and if a is half the Circuit of X , and b its Apotome, and c half the Circuit of Z , and d its Apotome, $ab = X$, and $cd = Z$; now ab is to cd in Ratio duple that of a to c , or that of b to d , † which by the Hypothesis is equal thereto, the Figures being Similar.

CORALLARY. THEREFORE the Surface of one Circle is to that of another, in a duple Ratio of its Circumference, or of its Diameter. ⁹¹

THEOR. 23. SIMILAR and regular Polygons, are one to another as the Squares of the Diameters of their circumscribing Circles. ⁹²
Eucl. 12. Prop. 1.

LET the two inscribed Polygons be X and Z . It must be demonstrated that they are one to another as the Squares of their Diameters AD , GK . Seeing that X and Z are similar Polygons, they are one to another in duple Ratio of the Side AB to the Side GH , by the preceding Theorem; but on account of their Similitude, the Triangles ABD , GHK are also Similar; for their Angles are all equal, being placed upon similar Arcs, † the homologous Sides are therefore Proportional, § and therefore $AB. GH :: AD. GK$; hence the duple Ratio of AB to GH is the same as the duple Ratio of AD to GK , therefore the Polygon X is to the Polygon Z , in duple Ratio of the Diameter AD to GK , that is, as the Square of AD to the Square of GK ; ||
Q. E. D.

CORALLARY. THEREFORE seeing that Circles may be taken ⁹³ for Polygons, their Surfaces are one to another as the Squares of their Diameters. Eucl. 12. Prop. 2.

Q 2.

Theorem

- | | | |
|-----------------|-----------------|----------------|
| * B. 2. n. 145. | † B. 2. n. 135. | ‡ B. 3. n. 54. |
| § Sup. n. 48. | Sup. n. 87. | |
| * B. 2. n. 145. | † Sup. n. 71. | ‡ B. 2. n. 39. |
| § Sup. n. 10. | B. 3. n. 80. | |

94 THEOREM 24. $AB=BC$; the Parallelogram $ABDH$ is greater than the Rectangle $AFGK$, and than any other whose point G is in the Diagonal DC . Eucl. 6. Prop. 27.

$BG=GE$, * then $BG+GC=GE+GC$, that is, that $BI=KE$. Now $BF=BI$, since that $AB=BC$; then $BF+BG$, or $AG=BG+KE$; now $BG+KE$ is lesser than BE , or its equal AD : therefore AG is lesser than AD .

95 PROB. 6. To apply to a given Line AB , a Parallelogram equal to C , wanting a Parallelogram similar to D . C ought not to be greater than a Parallelogram made Similar to D , and applied to the half of AB , that is, greater than AF or EG : Eucl. 6. Prop. 28.

DIVIDE AB equally at the Point E , upon EB describe the Parallelogram EG similar to D , † which according to what is required; will be greater than C , let the Figure I be the Excess thereof, let the Parallelogram NT be made equal to I , and similar to D or EG by the Problem, † let there be drawn the Diagonal FB , and make $FO=KN$, and $FQ=KT$, thro' O and Q draw the Parallels SR and QZ : then is AP the Parallelogram required. For the Parallelograms D, EG, OQ, NT, ZR are Similar one to another, and $EG=NT+C=OQ+C$; consequently C is equal to the Gnomon $OPQ=AO+PG=AO+EP=AP$; Q. E. D.

96 PROB. 7. To apply to a given right Line AB , a Parallelogram equal to C , and exceeding it by a Parallelogram similar to D . Eucl. 6. Prop. 29.

LET AB be equally divided in E , upon EB describe EG similar to D ; § then by means of the former Problems, || describe HK equal to $EG+C$ and similar to D or to EG , prolong FE and FG , so that $FL=IH$ and $FM=IK$; thro' L and M draw the Parallels RN, MN and AR ; prolong AB and GB ; draw the Diameter FBN , and then AN is the Parallelogram required. For these four Parallelograms D, HK, LM, EG are Similar by construction; therefore OP is similar to LM , or to D . * So likewise $LM=HK$, by construction; hence $LM=HK=EG+C$; then, $LM=EG+C$; then $LM-EG=C$. Now $LM-EG$ is equal to $BM+LB+BN$, or equal to $BL+AL+BN$; for $AE=EB$ by the Hypothesis; therefore $C=BL+AL+BN$, or $C=AN$; Q. E. D.

PROB. 8. A right lined Figure being given, to find another Similar thereto and in a given Ratio therewith.

97 LET the given Figure be A , 'tis required to find X , to which the Surface A hath a given Ratio, as for Example of 5 to 1: These Figures being supposed Similar, they ought to be one to another

* B. 2. n. 132. † Sup. n. 85. ‡ Sup. n. 86.

§ Sup. n. 85. || B. 2. n. 139. and Sup. n. 86.

* Sup. n. 51.

another as the Squares of their homologous Sides, * *that is, in duple Ratio of the said Sides, † or the same of some other homologous Line, Chord or Diameter which joins the Angles as B and C, therefore to do this, find such a mean Proportional C; as may be as 5 to 1; ‡ between the Diameter B and another Line D, so that $\therefore B. C. D.$ and upon the Line C describe the Figure X, similar to A, § these two Figures A and X are one to another as the Squares of B and C, || or in duple Ratio of B to C, that is, as to B to D, or 5 to 1; which was to be done.*

COROLLARY. *It is therefore easy to find a Figure similar to a given one, and less than another by a certain Figure given.* 98

By taking such another Line D as may bear the required Proportion to the Line B, the Side of the given Figure, and between them find as has been taught, * a mean Proportional C, and upon the same as a Side describe a Figure similar to the given one, † which by the Problem, will be the Figure required.

COROLLARY 2. *It is equally easy by the preceding Construction, to find a Figure similar to a given one, exceeding it by a certain Figure given.* 99

SECT. V.

OF the Commensurability, or Incommensurability of Lines and Surfaces.

DEFINITION I.

QUANTITIES are called Commensurable, when they can be measured by a Third, which is therefore their common measure. 100

THE common Measure of a Yard and an Ell, is an Inch, which is contained exactly thirty six times in a Yard, and forty five times in an Ell, meaning the Ell English.

DEF. 2. THOSE Quantities which cannot be measured by any common Measure, are Incommensurable. 101

A length of seven Ells cannot be exactly measured by a Yard; but it may be measured by a lesser Measure, as by a Foot and by an Inch. Hence two Quantities are incommensurable only when they cannot be exactly measured, by any measure how little soever it be.

COROLLARY. THEREFORE two Quantities having an infinite Proportion, are incommensurable to each other. 102

For

* Sup. n. 73. † B. 3. n. 80. ‡ Sup. n. 29.

§ Sup. n. 95. || Sup. n. 73.

* Sup. n. 29. † Sup. n. 8.

For if they were Commensurable, their common measure wou'd determine their Proportion, there is an infinite Proportion between a Point and a Line: for it cannot be determined in any Line whatsoever, how many Points are therein, seeing that it may be divided *ad inf.* So likewise the Proportion between a Line and a Surface is infinite: for in a Surface may be conceived as many Lines you please, so you may of Surfaces in a Solid.

103 DEF. 3. A rational Quantity is that whose Proportion to another is known and determined, and which is expressible by Number.

104 DEF. 4. Two Quantities which are not commensurable in themselves, are commensurable in Power, if their Squares or their Cubes are commensurable.

If b and c are Incommensurable, but that their Squares are Commensurable, as for Example let bb , be to cc as 3 to 5; then b and c Incommensurable in themselves, are Commensurable in the second Power. If x and z are not Commensurable nor their Squares xx and zz , but only their Cubes xxx and zzz , or x^3 , z^3 , for Example that if x^3 be to z^3 as 10 to 13, then x and z Incommensurable in themselves and in the second Power, are Commensurable in the third power.

105 DEF. 5. A square Number, is the Product of a Number multiplied by itself.

Thus 9 is a square Number, because 'tis the Product of 3 multiplied by 3, which is the Root thereof.

106 DEF. 6. A cube Number is the Product of a square Number multiplied by the Root of the Square.

Thus 8 is cube Number made by the square Number 4 multiplied by the square Root 2, that is, by 2, which multiplied by itself makes the Square 4.

107 DEF. 7. THOSE Numbers whose Square Roots are not expressible by Number, are not square Numbers.

108 DEF. 8. THOSE Numbers whose cube Roots cannot be expressed by Number, are not cube Numbers.

2, 3, 5, 6, 7, 8, 10, are Numbers which have no square Root expressible by number, for no number can be found which multiplied by itself makes either 2, 3, 5, 6, 7, 8, or 10, &c. Nor is there any cube Number among them except the 8, for there is not any number, which Cubically multiplied, can produce them.

109 DEF. 9. THE exponential Numbers of a Ratio, are the least Numbers which express it.

Thus if x is to z , as 6 to 12, the Exponents of the Ratio of x to z are 1 and 2, which are the least Numbers that can be between them, as 6 and 12.

WHEN two incommensurable Quantities are joined together, and which consequently cannot have the same Name, or which must be expressed by two different Names, this is called a Binomial.

When

When from one Quantity another which is Incommensurable thereto is subtracted, and that it cannot therefore be expressed with only one Sign, 'tis also a *Binomial*. But to distinguish this from that which is made by the Sum of two incommensurable Quantities, 'tis called an *Apotome*, *Residual* or diminished Quantity.

Evident Propositions.

PROPOSITION 1.

TWO Numbers are always commensurable one to another, for they have Unit (at least) for their common Measure. 110

For Example, in these two Numbers 77 and 81, as in all others, the Unit is contained exactly a certain Number of Times, therefore it is their common measure.

PROP. 2. WHEN one Number is subtracted from another, the Remainder is a Number. 111

For the Remainder is one or more Units.

PROP. 3. LINES and Surfaces which are as Number to Number, are commensurable; and those which are Incommensurable are not as Number to Number. 112

PROP. 4. TWO Quantities commensurable to a Third, are commensurable to each other. 113

For if B and Z are commensurable, their Ratio may be expressed by Number. If C and Z are likewise commensurable, their Ratio is also expressible by number; therefore these three Quantities may be all denoted by number; they are therefore commensurable.

PROP. 5. A Line, root or side of a Square which is not a Square number, is Irrational, if its Square were equal to a Square Number, it would be Rational. 114

THE Root of an irrational Number cannot be expressed by any Number whatsoever; therefore every Line equal to that Root, is inexpressible by number.

PROP. 6. If the Squares of two Lines are not one to another as two square Numbers, the two Lines are incommensurable. 115

THE Lines are each equal to the Root of an irrational Number, therefore inexpressible by Number, they are then incommensurable.

PROP. 7. A rational Square cannot be equal to two Squares, one of which is rational and the other Irrational. 116

THIS is what was said Proposition the Second; * that in taking one Number from another, the Remainder is a Number. Thus

* Sup. n. 111.

if from a rational Square, *that is*, which is a Number, be taken a rational Square or a Number, the Remainder ought to be a rational Square or Number.

- 117 PROP. 8. A commensurable Line being divided into two Parts, if one is commensurable, the other is so likewise.

For this Line is expressible by number, from which having taken away a commensurable part, that is, a Number, the Remainder will be a Number, according to the second Proposition.

- 118 PROP. 9. If two commensurable Lines are added together, their Sum will be Commensurable.

This is evident, a Number added to a Number makes a Number.

- 119 PROP. 10. If from a commensurable Line be subtracted an incommensurable one, the Remainder is Incommensurable.

For if one was Commensurable, the other would be so likewise, according to the eighth Proposition.

- 120 PROP. 11. FOUR Lines being in Proportion, if the First is commensurable to the Second, the Third will be so to the Fourth. If the First is Incommensurable to the Second, the Third will be so to the Fourth.

THIS is as much as to say, that if the Ratio of the First to the Second is expressible by number; that of the Third to the Fourth, which is the same, is also expressible by Number. If the first Ratio is inexpressible, it is the same of the Second.

- 121 PROP. 12. THE Square of a rational Line, is a Square Number; its Cube is also a Cube Number.

- 122 THEOREM 1. If simple Ratio's are of Number to Number, the Exponents of the duple Ratio's which are compounded thereof, are Square Numbers, and those of the triple Ratio's Cube Numbers.

LET the simple Ratio be that of b to d ; then the Ratio of b^2 to d^2 is Duple, now b and d being expressible by Number their Squares are so likewise. Hence b^2 and d^2 are equal to square Numbers, therefore b^2 and d^2 are one to another as square Numbers. By the same Reason b^3 and d^3 are one to another as Cube Numbers.

- 123 THEOR. 2. A simple Ratio is Surd, if its duple or triple Ratio be Surd.

FOR by the preceding Theorem, if the Ratio was not Surd, the Exponent of the duple Ratio would be a Square Number, and that of the triple Ratio a Cube Number.

- 124 THEOR. 3. A simple Ratio is not Surd, if the Exponential Numbers of the duple Ratio are Squares, and the Exponents of the triple Ratio Cubes, and if they be not so, it is a surd Ratio.

FOR the Squares are in duple Ratio of their Sides or Roots, and the Cube Numbers triple of their Root; hence the Roots of the Exponents of a duple or triple Ratio, are the Exponents of the simple Ratio; which consequently is not surd, by the same Reason if the Numbers are neither Squares nor

Cubes;

Cubes; as their Roots are the Exponents of simple Ratio's, these simple Ratio's not being expressible by Number, they are surd Ratio's.

LEMMA. FOUR Lines being in Proportion, the Product of the Antecedents is to that of the Consequents, as the Square of the first Term is to the Square of the second Term. ¹²⁵

LET them be $a. b::c. d$, it must be proved that $ac. bd::aa. bb$, the Product bc of the Means divided by the first Term is equal to the Fourth; * thus $\frac{bc}{a}$ may be put for d ; therefore $\frac{bbc}{a} = bd$:

Hence it must be demonstrated that $ac. \frac{bbc}{a} :: aa. bb$. Multiply

ac and $\frac{bbc}{a}$ by a , the Product is aac and bbc , and divide these products by c , there will remain aa and bb , the same Ratio will always remain; † therefore $ac. bd::aa. bb$, which was to be proved.

THEOR. 4. FOUR commensurable Lines being in proportion, the Rectangle or Product of the Antecedents, is to that of the Consequents, as two square numbers. ¹²⁶

LET the four commensurable and proportional Lines be $a. b::c. d$; by the preceding Lemma, $ac. bd::aa. bb$. Now a and b are supposed Commensurable; their Squares are therefore two square Numbers, therefore ac and bd are one to another, as two square Numbers.

LEMMA. SIX Lines being Proportional, the Product of the Antecedents is to that of the Consequents, as the Cube of the first Term is to the Cube of the second Term. ¹²⁷

LET them be $a. b::c. d::e. f$. It must be proved that $ace. bdf::aaa. bbb$; as $\frac{bc}{a} = d$, and $\frac{bc}{a} = f$: † Then $bdf = \frac{b^3ce}{aa}$.

Now in multiplying ace and $\frac{b^3ce}{aa}$ by aa , and dividing the Products of that multiplication $aaace$ and $bbbce$ by ce , the Quotients will be a^3, b^3 , the same Ratio will always remain; § therefore $ace. bdf::a^3. b^3$. which was to be proved.

THEOR. 5. SIX commensurable Lines being in proportion, the Product of the Antecedents is to that of the Consequents, as two cube Numbers. ¹²⁸

LET them be $a. b::c. d::e. f$; by the preceding Lemma, $ace. bdf::aaa. bbb$. Now a and b being Commensurable, are therefore expressible by Numbers, || a^3 and b^3 are two cube Numbers; consequently ace and bdf are one to another as two Cube numbers.

R

Theorem

* B. 3. n. 60. † B. 3. n. 54 and 55. ‡ B. 3. n. 60.
§ B. 3. n. 54 and 55. || Sup. n. 103.

129 THEOR. 6. If three Lines are in continued Proportion, and that the First is to the Third as two square Numbers, the three Lines are Commensurable.

LET the continued Proportion be $\div\div a. b. c$, if the Ratio of a to c is a square Number; I say that the Ratio of a to b will be a Ratio of Number to Number; for the Ratio of a to c is duple that of a to b , or as the Square of a to the Square of b ; * and consequently by Theorem 3. † this duple Ratio being as two square Numbers, the simple Ratio of which it is composed is Rational.

130 COROLLARY. In this Case the Rectangle of the Extremes and the Sides are Commensurable with the Square of the mean Term, and its Side.

FOR 1. Seeing that by the Supposition $\div\div a. b. c$, it will follow that $ac = bb$; ‡ and therefore ac commensurable with bb . 2. By the present Theorem, the Sides $a. b. c$, are commensurable.

131 THEOR. 7. If three Lines are Proportional, and that the First is to the Third as two Numbers, whose Product is not a square Number (or which have for Exponents two surd Numbers) the mean Term will be Incommensurable in itself, and Commensurable in power with the First and Third.

LET $\div\div b. c. d$, the First of the three Lines b is to the Third d , as these Numbers 2 and 9, whose Product 18 is not a square Number, I say that the second Line c is incommensurable in itself, and Commensurable in power with the First and Third. Its Square cc is equal to bd , or to 18. § Now 18 not being a square Number, its root c cannot be expressed by Number, therefore c is Incommensurable in itself with b and d ; but its power cc , which makes 18, is Commensurable with b and d ; that is, with 2 and with 9, seeing that its Ratio is expressible by number.

132 THEOR. 8. If of three Lines in continued Proportion, the First is not to the Third as Number to Number, the mean Term is Incommensurable with them, either in itself or in Power.

LET the Proportion be $\div\div b. c. d$, whose first Term b is not to d as Number to Number; I say that neither c nor its power cc are Commensurable with b and d ; for the Ratio of b to d is duple that of b to c , and of c to d . || this duple Ratio being therefore Surd, the simple Ratio of b to c , or of c to d is therefore Surd by the second Theorem. ¶ The Square of b is to the Square of c as b is to d . † Then seeing that b is not to d as Number to Number, bb and cc are not as Number to Number; therefore they are Incommensurable. ‡

Theorem

* B. 3. n. 86. † Sup. n. 124. ‡ B. 3. n. 157.

§ B. 3. n. 57. || B. 3. n. 83.

¶ Sup. n. 123. † B. 3. n. 84. ‡ Sup. n. 112.

THEOR. 9. *If of four Lines in continued Proportion, the Ratio of the First to the Fourth be a Ratio of Number to Number, having for Exponents Cube numbers, the four Quantities are Commensurable.*

LET $\div\div b. c. d. f.$ If $b. f::1. 8$, the four Lines b, c, d, f , are Commensurable; for the Ratio of b to f is triple. ¶ The two Numbers 1 and 8 are Cubes; this Ratio having therefore those cube Numbers for Exponents, the simple Ratio, as that of this Progression, which is between each Term, cannot be irrational, † according to Theorem the third.

THEOR. 10. *If of four Lines in continued Proportion, the Exponents of the Ratio of the First to the Fourth are not Cube numbers, the First and Second are Commensurable only in Power; it is the same likewise of the Second and Third.*

LET $\div\div b. c. d. f.$ and that $b. f::1. 5$. the Ratio of b to f is triple that of b to c . † Now 1 and 5, the Exponents of the Ratio of b to f , are not Cubes; therefore the simple Ratio of b to c is Surd: § So likewise of the Ratio of c to d , and that of d to f . But seeing that $b^3. c^3::b. f$; and consequently $b^3. c^3::1. 5$ ¶ therefore b and c are commensurable in power.

THEOREM 11. *If of four Lines in continued Proportion, the First is not to the Fourth as Number to Number, the Ratio of the First to the Second is not of Number to Number.*

$\div\div b. c. d. f$; the Ratio of b to f is triple; therefore this triple Ratio being surd, by the second Theorem, the simple Ratio of b to c is surd; and since that $b^3. c^3::b. f$, this Ratio of b to f being Surd, that of b^3 to c^3 is Surd. Therefore c is not Commensurable in power with b .

PROBLEM 1. *To find a Line which is incommensurable in itself, and Commensurable in power with a known or given Line.*

LET b be a Line, to find another Incommensurable therewith. The Line b being 2, take a Line x equal to 3 or to 5, or to any Number which is not a Square. Between these two Lines find a mean Proportional, which I call c : Hence $\div\div b. c. x$. Now by the seventh Theorem the Line c is Incommensurable in itself with the two first Lines, and commensurable in power. ¶

PROB. 2. *To find a Line that is Incommensurable, as well in itself as in power, with a known and given Line.*

LET the given and known Line be B , by the preceeding Problem find the Line D , which is in itself Incommensurable thereto. Then, between B and D , having found the mean Proportional Line C ; this Line by Theorem the eighth † will be Incommensurable with B , either in itself or in power; which was to be done.

R 2

Theorem

¶ $B. 3. n. 83.$ † $Sup. n. 124.$ † $B. 3. n. 83.$

§ $Sup. n. 124.$ ¶ $B. 3. n. 87.$

¶ $Sup. n. 134.$ † $Sup. n. 135.$

138. THEOREM 12. *THE Diagonal of a Square is Incommensurable in itself, and Commensurable in Power with each of the Sides.*

LET the Square be ABCD. It must be proved that its Diagonal AC, is incommensurable in itself with AB; but that it is Commensurable therewith in Power. LET $AB = a$, since that BC

$= AB$: Then $BC = a$: Therefore $aa + aa = AC^2$; then aa . $AC :: 1. 2$; thus you see the Diagonal is Commensurable in Power. Now 2 is not a square Number: † Therefore AC, the Root

of AC cannot be expressed by Number; ‡ it is therefore Incommensurable in itself; § which was to be proved.

139. THEOR. 13. *A rational Line being divided into mean and extreme Ratio, the two Parts thereof are irrational. Eucl. 13. Prop. 6.*

LET CB be a rational Line divided into mean and extreme Ratio at the Point A; I say that the Parts AC and AB are not rational Lines; or, which is the same Thing, that they cannot be expressed by Numbers. To CB join the Line BD, the half of CB, the Square of the intermediate Part AB joined with BD, makes five times the Square of BD; || therefore these two Squares are as 5 to 1. Now 5 is not a square Number: Therefore by the Theorem, ¶ the Line $AB + BD$ is irrational, but BD the half of the rational Line AB, is rational; the intermediate Part AB, must therefore be Irrational; † and therefore the lesser Part AC is Incommensurable; for if it were commensurable, AB would be so likewise. ‡

140. COROLLARY. *THE mean Part is Incommensurable with the whole Line, either in itself or in Power.*

LET b , be a Line divided into mean and extreme Ratio, x is the mean Part, and $b - x$ the least Part. $\div \div b. x. b - x$. Now since $b - x$ is by the present Theorem an irrational Line, then by the eighth Theorem § x , the mean Proportional between b and $b - x$, is incommensurable with b , either in itself or in power,

* Sup. n. 78.

† Sup. n. 106.

‡ Sup. n. 114.

§ Sup. n. 115.

|| Sup. n. 66.

¶ Sup. n. 114.

† Sup. n. 119.

‡ Sup. n. 117.

§ Sup. n. 132.

SECT.

S E C T. VI.

Of the Ratio's between the Chords and Radii of the Circle.

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THE Chords of one and the same Circle are not one to¹⁴¹ another as the Arcs whose Chords they are: The two Arcs BED and CFD being equal, the Arc BDC is double of either of them. If BC, the Chord of that Arc, was therefore double the Chord BD or of the Chord DC, as the Arc BDC is double the Arc BED, then BC would be equal to $BD + CD$; which cannot be. * Therefore it cannot be supposed that the Chords of one and the same Circle, are one to another as the Arcs which they are the Chords of.

THEOR. 1. *The Radius of a Circle is equal to the Chord of*¹⁴²
sixty Degrees.

THE Angle at the Center of an Hexagon is 60 Degrees, the sixth Part of 360 Degrees, which make the four right Angles that may be conceived about the Center. Then let BAC be an Angle of 60 Degrees; hence BC is the Side of the Hexagon, which must be proved equal to the Radius AB or AC. The Triangle BAC is Isocles; therefore the Angles at the Base are equal; but that of the Vertex is 60, therefore both together make 120, each therefore 60. The Triangle is then Equilateral; therefore $BC = AB$, or $BC = AC$; which was to be proved.

COROLLARY 1. *A Circle being given to describe an Hexagon.*¹⁴³
Eucl. 4. Prop. 15.

THE Compasses being opened to the Extent of the Radius, it will divide the Circle into six equal Parts.

COROLLARY 2. *It is easy to inscribe an equilateral Triangle in*¹⁴⁴
a Circle.

FOR two Parts of the Hexagon's six, make the third Part of the Circle.

COROLLARY 3. *The Ratio of the Circle's Circumference to its*¹⁴⁵
Radius, is above 3 to 1.

EACH Side of the Hexagon being equal to the Radius, the six Sides are equal to three times the diameter; hence the Circumference of the Polygon is to the Radius of its circumscribing Circle, as 3 to 1. Now the Circumference of the Circle is greater than that of the Hexagon, each Portion of the Circle being greater than the Side of the Hexagon, which is the Chord thereof.

Theorem

* B. 1. n. 12.

146 THEOREM 2. THE Square of one of the Sides of an equilateral Triangle inscribed in a Circle, is triple the Square of the Semi-diameter or Radius of the Circle. Eucl. 13. Prop. 12.

LET ABC be an equilateral Triangle inscribed in the Circle X; let there be drawn the Diameter AD which cuts the Chord BC perpendicularly into two equal parts. * It must be proved that the Square of AB is triple that of the Radius; since that BC is the Chord of the third Part of the Circle, BD is the Chord of the sixth Part, and therefore equal to the Radius, of which the Diameter AD is double. † hence if $BD=b$, then $AD=2b$, and

$AD^2=4bb$. Let $AB=a$; since that ABD is a right Angle: then

$aa+bb=AD^2$. ‡ Therefore $aa+bb=4bb$. Taking away bb from each Side, there will remain $aa=3bb$; Q. E. D.

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147 THERE are several such like Theorems, which I shall not give Place to in these Elements. It is evident that the Tangent of an Angle of 45 Degrees is equal to Radius: For it makes with the Radius a rectangled Triangle, whose Base is the Secant, the Angles at the Base are equal, (viz.) each of 45 Degrees: Therefore, the Tangent and the Radius are equal. 2. The Square of the Secant is equal to the Square of Radius and of the Tangent, † or which is the same, to twice the square of Radius. Now the Chord of ninety is equal to the Secant of forty-five Degrees: Consequently the Chord of ninety is equal to twice the Square of Radius. as is evident.

LEMMA 1. IF the Angles at the Base of an Isosceles Triangle, 148 are each double that of the Vertex, I say that the Line which divides one of the Angles at the Base into two equal Parts, will divide the opposite Side into mean and extream Ratio.

LET the Isosceles Triangle be BAC, and CD the Line which divides BCA, one of the Angles at the Base into two equal Parts; it must be proved that it divides AB into mean and extream Ratio at the Point D. 1. Seeing that the Angle BCA is supposed to be double of BAC; then the half DCA is equal to the Angle CAD: Wherefore the Triangle ADC having the Angles at the Base AC equal, it is Isosceles, § and the Sides AD and DC will be equal. 2. The Angle BDC is equal to the two opposite ones DAC and ACD; * consequently it is equal to the Angle ACB which make these two Angles, and to DBC which is by the Hypothesis equal to ACB; therefore the Triangle DCB having its Angles at the Base DB equal, is also Isosceles; Hence $DC=BC$. 3. The two Isosceles Triangles BAC and BDC, which have at the Point B an Angle common, † are equiangular, and therefore similar: Then ‡ AB is to BC, or to AD its equal, as DC or its

* B. 1. n. 90. † Sup. n. 142. ‡ Sup. n. 78. † Sup. n. 78.

§ B. 2. n. 83. * B. 2. n. 73. † B. 2. n. 85. † Sup. n. 10.

its equal AD is to DB, that is, $\div\div$ AB. AD. DB; and consequently AB is divided into mean and extream Ratio; since that the Part AD is a Mean between the whole AB, and the other Part DB. $\downarrow$

LEMMA 2. To describe an Isosceles Triangle, having each of its Angles at the Base double the other. Eucl. 4. Prop. 10. 149

HAVING the Line AB, divide it into mean and extream Ratio at the Point D; § from B and D, with the Extent of the mean Part AD, describe two Arcs which intersect at the Point C. Hence $BC=DC=AD$. The two Triangles ADC and DCB are therefore Isosceles. By the same Construction $\div\div$ AB. AD. DB. substituted in the Room of equal Lines AB. BC $\therefore$ BC. BD; then BAC and DBC are similar, or equiangular * and Isosceles. Now the Angle BDC is equal to $DAC+ACD$ the opposite interior Angles which are equal, seeing ADC is Isosceles: Therefore DBC equal to BDC is the Double of BAC; which was to be proved.

LEMMA 3. In an Isosceles Triangle, whose Vertex is at the Centre of the Circle, and its Base the Side of a Decagon, each Angle at the Base is double that at its Vertex. 150

THE Isosceles Triangle BAC hath its Vertex at the Centre of X, its Base BC is the Side of a Decagon, and consequently the Chord of an Arc of 36 Degrees the tenth Part of 360 Degrees which make the whole Circle. It must be proved that each Angle at the Base CBA and ACB, contains 72 degrees, the Double of 36; which is evident, for the three Angles make 180 Degrees, † if from this Number be taken 36, the Content of the vertical Angle, 144 remains for the Angles at the Base, which being equal, each is 72 the Double of 36.

THEOREM 3. THE Radius being divided into mean and extream Ratio, the mean Part is the Side of a Decagon, or Chord of thirty six Degrees. Eucl. 4. Prop. 10. (Fig. above) 151

LET AC be the Radius of a Circle divided into mean and extream Ratio at the Point D, † and that the Chord BC is equal to the mean Part AD, then the Angles at the Base BC are each double of BAC. § Therefore by the preceeding Lemma, BC is the Chord of the twelfth Part of the Circle, and consequently the Side of the Decagon.

LEMMA 4. Two Lines being drawn from one of the Angles of a Pentagon to the Ends of the opposite Side, they make an Isosceles Triangle, and each of the Angles at the Base is double the Angle at the Vertex. 152

Z is a regular Pentagon, having drawn from A two Lines to the Extremities of the opposite Side BC, they make the Triangle BAC, which must be proved to be Isosceles, and that ABC and BCA are each double of BAC. 1. AB and AC the Chords of equal

† Sup. n. 34.

§ Sup. n. 34.

* Sup. n. 13.

† B. 2. n. 75. † Sup. n. 34. § Sup. n. 149.

equal Arcs, are equal: Therefore BAC is Isocetes. * 2. The Angle BAC is measured by half the Arc BC, and the Angle ACB by half the Arc AEB. § Now AEB is double of BC: Therefore the Angle ACB is double the Angle BAC. It is in like manner proved that ABC, is double of BAC.

153 THEOREM 4. If in a Pentagon two lines intersect, each of which is the Chord of double the Arc subtended by each Side of the Pentagon, they are divided into mean and extreme Ratio, and the mean or longest Part is one of the Sides of the Pentagon.

SUPPOSE Z to be a regular Pentagon inscribed in a Circle, the Lines BD and EF intersect, each being the Chord of double the Arc subtended by each of the Sides of the Pentagon, it must be proved that they are cut into mean and extreme Ratio, and that the mean or longest Part, such as BC, is one of the Sides of the Pentagon. 1. The Angle BEF is measured by half the two Arcs BG and GF, † and the Angle BCE, by half the two Arcs BE and FD, ‡ Now these two Measures are equal: therefore the two Angles are equal; hence EBC is Isocetes, || and therefore BE = BC: Therefore BC is one of the Sides of the Pentagon; which was to be proved. 2. By the same Reason CF = FD: And since BD = FE are Chords of the same Angles, it must be that CE = CD: Therefore ECD is Isocetes. The Triangle BED is also Isocetes, seeing that BE = ED; for Z is supposed to be a regular Pentagon: The two Isocetes have one Angle common at D, therefore they are equiangular: * Wherefore as BD is to BE, or its equal BC; so ED or its Equal BC is to CD, § that is, $\therefore BD. BC. CD$; and consequently, according to the Definition of mean and extreme Ratio, BD is divided as was proposed.

154 THEOR. 5. THE right Line compounded of the Side of an Hexagon and that of a Decagon both inscribed in one and the same Circle, is divided into mean and extreme Ratio. Eucl. 13. Prop. 9.

LET the Side of the Decagon be AB, and the Side of the Hexagon BE, that is, a Line equal to the Radius BD; * I say that AE is divided into mean and extreme Ratio at the Point B. EBD is Isocetes, seeing that BE is supposed equal to the Radius BD; the Triangle ABD is also Isocetes, and the Angle BAD or ABD is the double of BDA. † Now DBA is also the double of BDE, as being equal to the two equal Angles BED and EDB: || therefore BED, and BDA are equal one to another, and taken together they are equal to EAD; hence the Triangle AED is Isocetes; wherefore, seeing that BD divides the Angle EDA into two equal Parts, EA is therefore divided into mean and extreme Ratio. * Q. E. D.

Corallary.

* B. 1. n. 31. & B. 2. n. 57. § B. 2. n. 39. † B. 2. n. 39.

‡ B. 2. n. 52. || B. 2. n. 83.

* B. 1. n. 85. § Sup. n. 10. * Sup. n. 142.

† Sup. n. 150. || B. 2. n. 73. * Sup. n. 148.

COROLLARY. FROM hence it follows that if AE is divided into mean and extreme Ratio, and that its Segment AB is the Side of a Decagon, BE will be the Side of an Hexagon inscribed in the same Circle.

SEEING that there can be but one mean Proportional between AE and AB , and it has been proved that BE , the Side of the Hexagon is that mean.

PROBLEM 1. A Circle being given, to find the Chord of 36 Degrees, or Side of a Decagon. See fig. To Num. 150.

THE Circle is X , and the Radius is AB , which divide into mean and extreme Ratio at D . * Take the Chord BC equal to AD , which is the Side of the Decagon required. †

PROB. 2. To describe a Decagon upon a Line given for a Side.

DIVIDE the given Side BC into mean and extreme Ratio, † and join thereto the mean Part; which gives a new Line AC , divided likewise into mean and extreme Ratio, whose mean part is BC ; † describe a Circle having AC for its Radius, then will BC be the Side of its inscribed Decagon; § whereby the whole may be easily constructed.

PROB. 3. A Circle being given, to find the Side of a Pentagon, or to Inscribe a Pentagon in a Circle. Eucl. 4. Prop. 11. 158

HAVING found the Side of the Decagon, ‖ that of the Pentagon is the Chord of double the Arc, subtended by one of the Sides of the Decagon. 2. The Side of the Pentagon may also be found thus. Describe an Isosceles Triangle, having its Angles at the Base each double that at the Vertex, * and either inscribe that (or one Similar thereto) in the given Circle, † which will be the Side of the Pentagon sought. ‡

PROB. 4. To describe a Pentagon upon a given Side. 159

LET the given Side be ED , suppose the Pentagon to be already made, then will the Method of finding the Lines BD and EF , discover what must be done. Now seeing that BD is divided into mean and extreme Ratio at the Point C , § that BC equal to BE , and consequently to ED , is the mean part, by dividing the Side ED into mean and extreme Ratio, and adding thereto the mean Part, there will arise a new Line equal to BD , ‖ divided likewise into mean and extreme Ratio, whose mean Part is ED , consequently equal to BC . Then as the three Sides of the Triangle DEB are known, it is easy to compleat the whole Pentagon, by circumscribing a Circle about the Triangle. *

PROB. 5. To Circumscribe a regular Pentagon about a Circle. Eucl. 4. Prop. 12. 160

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1. IN

* Sup. n. 34.	† Sup. n. 151.	‡ Sup. n. 34.
‡ Sup. n. 35.	§ Sup. n. 156.	
‖ Sup. n. 156.	* Sup. n. 149.	† B. 2. n. 100.
‡ Sup. n. 152.	§ Sup. n. 153.	‖ Sup. n. 35.
* B. 1. n. 87.		

1. In the given Circle inscribe a Pentagon. 2. Draw right Lines from the Center of the Circle thro' the five Angles of the Pentagon. 3. Draw Tangents thro' the Points of the Circumference where these Lines cross the Circle. These Tangents, as is evident, will form the Pentagon required.

161 PROB. 6. To inscribe a Circle, in a regular Pentagon. Eucl. 4. Prop. 13.

UPON two of the Sides of the Pentagon, raise Perpendiculars, the Point where they intersect is the Center of the Circle required, having for its Radius the Extent between the Center and the Middle of one of the Sides of the Pentagon, which is evident.

162 PROB. 7. To circumscribe a Circle about a Pentagon. Eucl. 4. Prop. 14.

HAVING by the preceeding Problem found the Center of the Circle, take the distance between the Center and one of the Angles of the Pentagon, a Circle described with that extent will be the Circle required.

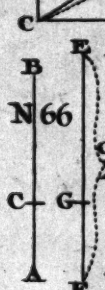
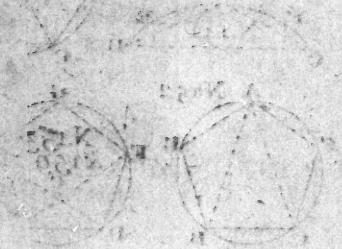
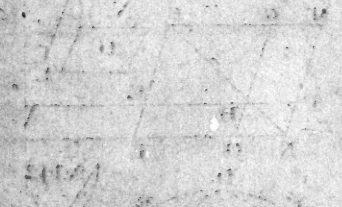
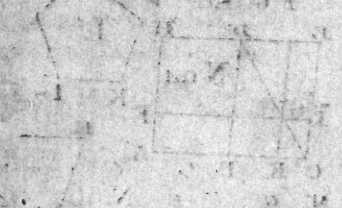
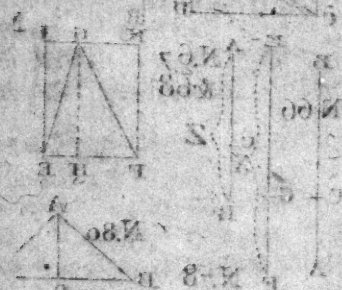
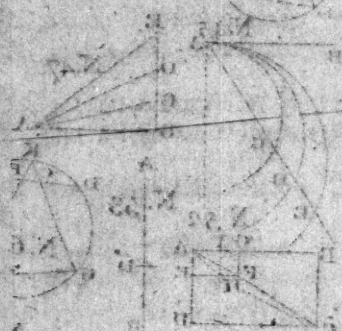
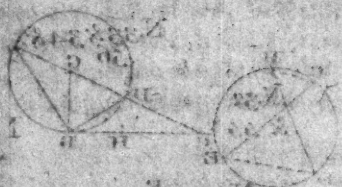
163 PROB. 8. To inscribe in a Circle a Quindecagon, or Polygon of Fifteen Sides. Eucl. 4. Prop. 16.

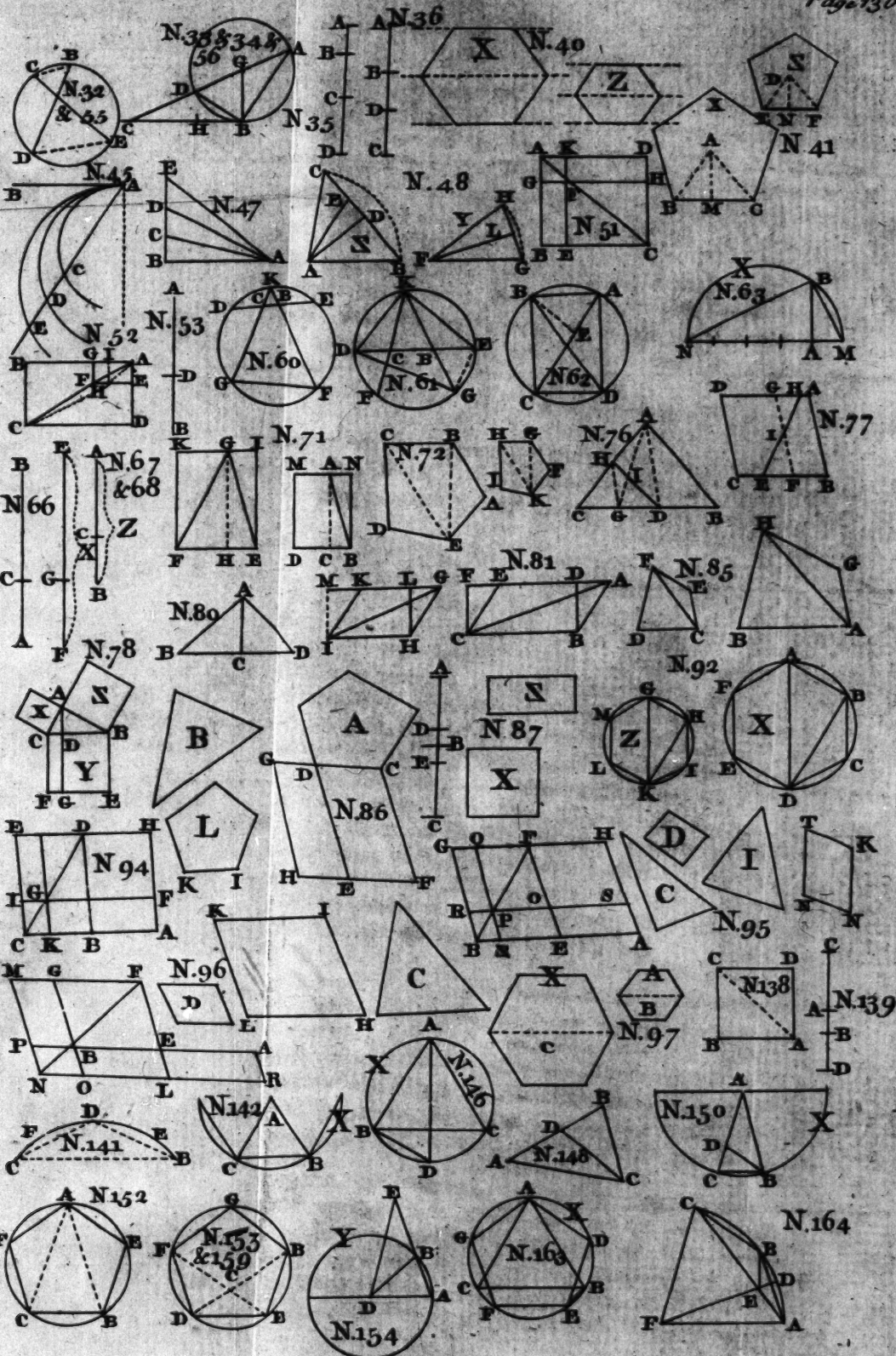
THE Circle X being given to inscribe therein a Quindecagon, 1. Divide it into six equal Arcs; the Chord of each of those Arcs is the Radius thereof. 2. Take two of these Arcs, the double whereof is the third Part; the Chord thereof is therefore a Side of the Equilateral Triangle ABC. 3. Make the Pentagon ADEFG, each Side of which is the Chord of the Arc of the Circle's fifth Part; the Chord of the third Part of this Arc, is the Side of the Quindecagon, five Sides of which make the third Part of the Circle, and consequently answer to the Side of the Equilateral Triangle. This being so, BE is the Side of the Quindecagon required; for three parts of the Polygon answer to AD, and as many to DE. Hence six Sides will answer to AD and DE: But five Parts answer to AB; therefore BE, the sixth Part of AE, remains for the Side of the Quindecagon.

164 THEOREM 6. THE Square of one of the Sides of a Pentagon is equal to the Squares of one of the Sides of a Decagon, and of a Hexagon, inscribed in the same Circle. Eucl. 13. Prop. 10.

LET AC be the Side of a Pentagon. Having divided the Arc AC into two equal parts at the Point B, and drawn the Chords AB, BC, they are the Sides of the Decagon, and AF or FC Radii of the Circle, are the Sides of the Hexagon: It must be

proved that $AC^2 = AF^2 + AB^2$. Let there be drawn the Line FD dividing AB into two equal Parts, and from the Point E let there be drawn EB; there will be the two Triangles ACF and ECF which are Similar; for they have the Angle C common; and the Angle CFE is equal to FAC; because they both contain an equal number of Degrees, viz. 54, for the Arc BC being of 36 Degrees





Degrees and BD of 18, DC will be of 54, the Measure of the Angle CFD, to which CAF is equal because the Triangle AFC is Ifoceles, and the Angle of the Center is 72 Degrees. Then these Triangles being Similar $\therefore AC:AF:EC$: * Therefore

$AF=AC \times EC$. † Now the Ifoceles Triangle AEB is Similar to ABC which is also Ifoceles; for they have one Angle common, viz. A, and consequently all the others equal. † Then $\therefore AE$.

$AB:AC$, then $AB=AE \times AC$; then $AF+AB=AC \times EC+AE \times AC$. But these two Rectangles are equal to the Square of

AC; § for $AE+EC=AC$. Therefore $AC=AF+AB$; Q. E. D.

COROLLARY. THE Square of the Side of a Pentagon inscribed in a Circle, and the Square of the Chord which subtends the Angle of the Polygon joined together, are equal to five times the Square of the Radius of the Circle. 165

LET BF the Side of the Pentagon be equal to b , and BC the Diameter of the Circle $=2a$, and BD the Subtense of the Angle

F of the Pentagon $=d$. It must be proved that $BF+BD=5AB$, or $bb+dd=5aa$. Let there be drawn $DC=c$, which will be the Side of the Decagon, and will form the right angled Triangle BDC. || Then $dd+cc=4aa$; * but by the preceding Theorem $bb=aa+cc$; this Equation added to the First, the Sum will be $bb+dd+cc=aa+cc+4aa$; subtract cc from each Side, $bb+dd=5aa$; which was to be proved.

PROB. 9. A Circle being given, to find the Side of a Pentagon and of a Decagon, by a Method different from the Former. 166

LET the Diameter of the given Circle be AB, and its Radius CD Perpendicular upon AB; if there be taken CE half of BC, and that from the Point E be drawn ED, and that afterwards be taken EF=ED and there be drawn DF; I say that this Line

will be the Side of the Pentagon required. For $BF \times FC+EC=EF=ED$. † But $FD=DC+FC$, † hence $BF \times FC+EC=DC+EC$; then taking away EC from each Side, $BF \times FC=DC=$

BC; and this Equation reduced to a Proportion, § it will be $\therefore BF:BC:FC$; therefore the Line FB is divided into mean and extreme Ratio; || and seeing BC or DC is the Side of the Hex-

agon, FC will be that of the Decagon. * But $FD=DC+CF$; † therefore

* Sup. n. 10. † B. 3. n. 57. † B. 2. n. 85.

§ B. 3. n. 18. || B. 2. n. 44.

* Sup. n. 78. † B. 2. n. 22. † Sup. n. 78.

§ B. 3. n. 58. || Sup. n. 34.

* Sup. n. 154. † Sup. n. 78.

therefore DF is the Side of the Pentagon; * which was to be found.

OR THUS

- 167 Take BC equal to the Radius of the Circle, BD equal to the Chord of Ninety, or to the Side of a Square inscribed in the Circle, and CE equal to BD. The Line DE will be the Side of the Pentagon, and AD being the Radius of the Circle, AE will be the Side of the Decagon. To demonstrate the Truth thereof, let fall from C a Perpendicular upon the Radius AB, and from D upon the Center A, the Perpendicular AD; hence BD is the Chord of Ninety, to which CE is equal. Suppose AB, AD or BC = $2a$; hence BG = $4a$. Now $\div$ BG. BC. BF, † that is, $\div 4a. 2a. a$; then BF is half the Radius. But it appears by the present Theorem, that to find ED the Side of the Pentagon, there is made FB = FD, if then it be proved that there is this Equality in the present Construction, it is a sufficient Demonstration. Since FB = a , FG will be = $3a$, the Rectangle GF $\times$ FB, that is, $3aa = FC$. ‡ Now AB and AD being each $2a$, their Squares together are $8aa = DB = CE$ by Construction, § equal to CF + EF. If then from CE = $8aa$ be taken FC = $3aa$, there will remain $5aa$ for EF, to which FD is also equal seeing it is equal to DA + AF, || that is, to $4aa + aa$. Wherefore FD = FE; ¶ E. D.

- 168 THEOR. 7. If three Angles of an Equilateral Pentagon being taken any how are Equal, it is Equiangular. Eucl. 13. Prop. 7.

LET the Equilateral Pentagon be ABCDE, which hath three of its Angles equal, take them how you will, viz. A. C. D; it must be proved that they are all equal, that C = B = E. The Triangles BAE, BCD CDE being Isosceles, whose equal Angles A, C, D are comprehended by equal Sides, they are every where equal; * and therefore the Bases BE, BD, CE are equal, as also all the Angles formed by equal Sides upon the said Bases are equal. So likewise the two Isosceles Triangles BEC, EBD having all their Sides equal, they have also the Angles EBC, ECB, BED, BDE equal. † If to each of the equal Angles EBC, ECB the equal ones ABE, DCE be added, the Total B will be equal to the Total C. The same for the Angles E and D, therefore the five Angles are all equal; which was to be proved.

Theorem.

* Sup. n. 164. † Sup. n. 18. ‡ Sup. n. 28 & B. 3. n. 57.

§ Sup. n. 78.

|| Sup. n. 78.

* B. 2. n. 98.

† B. 2. n. 92.

THEOREM 8. WHEN the Radius of a Circle is Rational, the Side of a Decagon inscribed in the Circle is incommensurable with the Radius, both in itself and in power. 169

LET B the Radius of a Circle be a rational Line, divided into mean and extreme Ratio, x which I suppose to be the mean Part, will be the Side of the Decagon. * Now the mean Part x is incommensurable with B , both in itself and in power. †

THEOREM 9. WHEN the Radius of a Circle is rational, the Sides of the Pentagon inscribed in the Circle are Incommensurable with the Radius, either in themselves or in power. Eucl. 13. Prop. 11. 170

LET the rational Line b , be the Radius of a Circle, x the Side of a Pentagon, and z the Side of a Decagon, both inscribed in the Circle whose Radius is b . Therefore $bb + xx = zz$: † then seeing that the Square xx is Irrational, as we demonstrated in the last Theorem, the Square zz cannot be rational, § consequently its root z is Irrational, since that every Square which is not equal to a Square Number, cannot have for its Root a Quantity exactly equal to any Number, as we have proved above. ||

OF the Quadrature of the Circle.

WE have shewn * that the Surface of the Circle is equal to a Triangle, having for its height the Radius of the Circle, and the Circumference thereof for its Base; therefore a mean Proportional between the Radius and the Circumference, is the Side of a Square equal to the Circle. But to find this mean Proportional, the Radius being given, a right Line must be found equal to the Circumference, which cannot be done without knowing the Ratio of the Radius to the Circumference, which is known nearly, but not exactly. The Circle may be taken for a Polygon whose Sides are infinite; therefore the most exact and least deficient Method of finding the said Ratio, is to take the greatest Polygon possible; and to see what is the Ratio of its Circumference with the Diameter of its circumscribing Circle. * Archimedes considered a Polygon of Ninety six Sides, the Circumference of which to the Diameter of the Circle is as 223 to 71, which Ratio is less than that of the Circumference of the Circle to the Diameter, the inscribed Polygon being less than the Circle. † Comparing the same Polygon, but considered as a Circumscribing one with the Diameter, he found that the Ratio of one to the other was as 22 to 7, which is greater than that of the Circle's Circumference to its Diameter; by reason a circumscribing Polygon is greater than the Circle. Here

* Sup. n. 151. † Sup. n. 140. ‡ Sup. n. 164.

§ Sup. n. 116. || Sup. n. 114.

* B. 2. n. 154. † B. 3. n. 147.

Here is therefore two Ratio's (viz.) that of 223 to 71, and that of 22 to 7, one of which is lesser, and the other greater than the true one required. Reduce these two Ratio's to one and the same Consequent. *

$\left. \begin{matrix} 1561 \\ 1562 \end{matrix} \right\} 497$. HAVING therefore supposed the Diameter of 497 Parts, the Circle's Circumference will be greater than 1561, and less than 1562. Dividing unity which is the difference of these two numbers by 497, the Quotient $\frac{1}{497}$ shews that the Difference which each number differs from the true Length of the Circumference, is less than $\frac{1}{497}$ part of the Diameter; which is but a small matter.

Those who have taken Polygons containing more than 96 Sides, have found an exacter Ratio. Pell considered one of 256 Sides, each of which is less than 24545, supposing the Radius of the Circle to be of 1000000; therefore the Semi-circumference of the Polygon is less than 3141760. If the Diameter of the Circle was therefore 1000000, the whole Circumference of the circumscribed Polygon would be to the Circle less than 314176. Now the Calculus contained in a Work printed at Amsterdam in the Year 1647, supposes no extraction of the Root; and is grounded upon the following Theorem.

THEOREM.

BC being the Tangent of an Arc less than forty five Degrees, and BD the Tangent of a double Arc; BC is to BD as the Square of the Radius AB, less the Square of BC, is to twice the Square of the Radius AB. It must be demonstrated that $BC : BD :: AB^2 - BC^2 : 2AB^2$

Seeing the Angle BAD is supposed double BAC; then $AB : AD :: BC : CD$, † and compoundedly, $AB : AD + AB$

(or DE) :: $BC : BD$. ‡ Wherefore $AB : DE :: BC : BD$. § Alternately, $AB : BC :: DE : BD$; then dividedly, and compoundedly, $AB - BC : 2AB :: DE - BD : 2DE$.

Hence it remains to be demonstrated that $DE - BD : 2DE :: BC : BD$, for then $AB - BC : 2AB :: BC : BD$.

Seeing that two Ratio's equal to a Third are equal one to another. Now $DE = AE + 2AE \times AD + AD^2$. ||

But $AD = AB$ (or $AE + BD$.) then $DE = 2AE + 2AE \times AD + BD^2$. Then

* B. 3. n. 89.

† Sup. n. 17.

‡ B. 3. n. 49.

§ B. 3. n. 84.

|| B. 3. n. 20.

* B. 3. n. 140.

Then subtracting $\overline{BD}$ from each Side $\overline{DE} - \overline{BD} = 2\overline{AE} + 2\overline{AE} \times \overline{AD}$.
 Now $2\overline{AE} + 2\overline{AE} \times \overline{AD} = 2\overline{AE} \times \overline{ED}$: * Then $\overline{DE} - \overline{BD}$
 $= 2\overline{AE} \times \overline{ED}$. These two Rectangles $2\overline{AE} \times \overline{ED}$ and $2\overline{ED}$ have
 the same Bale, viz. $\overline{ED}$; they are therefore one to another as
 $\overline{AE}$ or $\overline{AB}$ is to $\overline{ED}$, Now $\overline{AB} : \overline{ED} :: \overline{BC} : \overline{BD}$; therefore $\overline{DE}$
 $- \overline{BD}$ equal to $2\overline{AE} \times \overline{ED}$ is to $2\overline{ED}$ as $\overline{BC}$ to $\overline{BD}$; which was
 to be shewn.

According to the Archimedian Proportion of the Circumference of the Circle to its Diameter, viz. That of 22 to 7, or 44 to 14, or 66 to 21, which are each the same, the Surface of the Circle is to the Square of its Diameter as 11 to 14. For let the Circumference of the Circle be called m , and the Diameter n , multiply m and n by n , then mn . nn . m . n . † and seeing that $m : n :: 44 : 14$; then $mn : nn :: 44 : 14$, therefore the fourth Part of mn is to nn as 11, the fourth Part of 44, is to 14. Now the Quarter of mn is the Surface of the Circle. ‡ this Surface is therefore to nn , the Square of the Diameter, as 11 to 14.

Hence the Quadrature of the Circle would have been found, 174 if the true Ratio of the Circumference to the Diameter had been, as 22 to 7; but this Ratio is not exact, and what is the true one is not yet known. Nevertheless this Quadrature is partly known, that is, a Portion of a Circle may be assigned equal to a rectilinear Figure; for the Triangle ABC being right Angled, the Semi-circle ADBE is equal to the two Semi-circles AGB and BFC. § then taking away the Parts which are common, viz. ABD and BCE, there will remain AGBDA and CEBF, which are equal to the Remainder of the Semi-circle ADBEC, which remainder is the Triangle ABC. It also appears to the Eye that the Figure AEBFCDGH, is equal to the Square ABCD; for by construction the Parts AHD = AEB = BFC = CGD.

The Figures themselves shew that a Triangle circumscribed about a Circle is Quadruple of the inscribed one, that a circumscribed Square is double the inscribed one, that the circumscribing Hexagon is to the inscribed one as 4 to 3. Observe here by the Way, that to divide a Line as BE Geometrically into three equal Parts, make the Side of an equilateral Triangle inscribed in a Circle, then having made the Hexagon and divided the Sides thereof by Perpendiculars, you will find that the Perpendiculars will divide BE into three equal parts BC, CD, DE.

The

* B. 3. n. 29.

† B. 3. n. 54.

‡ B. 2. n. 154.

§ Sup. n. 79.

The following is a general Theorem which affords a very extensive Idea, of what may be known in this matter.

- 176 WHEN of two inscribed Polygons one has twice as many Sides as the other, the Greater is to the Lesser, as the Radius of the Circle is to the Apotome of the lesser Polygon.

Z is a Polygon whose Apotheme is AB, and X a Polygon having twice the Number of Sides. The Surface of Z is equal to as many times the Triangle DAE, as Z hath Sides; and it is plain, that the Triangle DCE being added to each of these Triangles, will give the Surface of X, which is to that of Z, as the Rhombus AECD is to the Triangle DAE. Now these two Figures are one to another as the Radius AC is to AB, which is the Apotome of Z; for the Surfaces of the two Triangles DAE and DCE, which have the same Base, are one to another as their Heights AB and BC; so of all other Polygons, of which one has twice as many Sides as the other. If the Polygon X is divided so that it may have a double Number of Sides, which I call Y; it will be the same: And it is proper to observe, that the first Polygon Z, is to the Third Y, as the Rectangle of the Apotomes of the First and Second is to the Square of the Diameter; which I demonstrate thus. Let the Apotome of the First be called m ; that of the Second n ; and the Radius k , according to what has been demonstrated.

$$\begin{cases} Z. X :: m. k. \\ X. Y :: n. k. \end{cases}$$

The Antecedents being multiplied by the Antecedents, and the Consequents by the Consequents, $ZX. XY :: mn. kk$. Now $ZX. XY :: Z. Y$; then $Z. Y :: mn. kk$. The first Polygon Z is to the Third Y, as mn , the Rectangle of the Apotomes of the First and Second is to kk the Square of Radius.

- 177 The same Way it is demonstrated that the first Polygon is to the Fourth, as the Solid produced by the Apotomes of the First, Second and Third is to the Cube of the Radius of the Circle, and so of the Rest; *ad. inf.* The Apotome of an inscribed Square is the half of one of its Sides; hence a Square is to an Octagon as the half of one of its Sides is to the Radius of its circumscribing Circle; or which is the same, as one of the Sides is to the Diameter of the Circle. Wherefore it is justly concluded, that an inscribed Square is to a Polygon of infinite Sides, doubling always the Sides, *that is*, of a Square is made an Octagon, of an Octagon a Polygon of sixteen Sides, and so on which Polygon may be considered as a Circle; it may, I say, be concluded that the Square is to the Circle as a Quantity produced

produced by the Multiplication of all the Apotomes of the Polygons, to the highest Power of the Radius of the Circle. But this is knowing scarce any thing; for besides the surd Ratio's which we frequently meet with, the Radius and Diameter are divisible *ad. inf.* Therefore the Number of all the Apotomes cannot be comprehended. In fine it is therefore impossible by this way to reach from a Polygon to a Circle; this Figure is therefore incomprehensible, altho' it be the Simplest of all Geometrical Figures.



B O O K. V.

OF Solids, or the third Species of Extension.

S E C T. I.

OF the Sections and meeting of Planes.

ADVERTISEMENT.

EVERY Solid of what Sort soever it be, may be conceived as composed of different Planes placed one upon another; or as described by the Motion of one only, moving in a certain manner. Also according as two or more Planes intersect or meet, they form Solids; which obliges us in treating hereof, to begin with the meeting and Section of Planes.

N. B. *The References to the paste board Plates annexed to this Work are denoted by the Letters P. B.*

EVIDENT Propositions concerning Planes.

PROPOSITION 1. *A right Line in moving along another right Line, which is at rest, and keeping with it one and the same Position, describes a superficial Plane, or simply a Plane.*

THIS Superficies hath all its parts equally placed between its Extremities; for they are uniformly described: And that which makes the Difference of other Superficies, which are not Planes, is the Way by which they are generated, a right Line may be every way applied to it, as is going to be declared.

PROP. 2. *A right Line may be every way applied to a plane Surface, and will agree with it.*

PROP. 3. *THE Surface of a Plane is the Shortest that can be conceiv'd between the Ends or Bounds thereof.*

FOR a Plane is composed of right Lines, which are the Shortest that can be conceived between their Extremities.

PROP. 4. *A Plane raised upon a Plane is Perpendicular thereupon, when it inclines no more to one Side thereof than to the other.*

PROP. 5. *TWO Planes are Parallel, when all their Parts are at an equal Distance one from the other, and which being produced will never meet.*

P. op. 6.

PROP. 6. *A Plane may be produced, or conceived as produced from all its Sides, as much as shall be requisite.* 6

PROP. 7. *A right Line cannot be part upon a Plane, and part above it.* Eucl. 11. Prop. 1. 7

FOR if it were, then this Line cannot be applied to the Plane and agree with it, therefore according to the second Proposition, the Plane would not be a Plane. Besides this, it may be conceived that the Part of this Line which is on the Plane being prolonged, will always remain on the Plane, is therefore a different Line from that which is out of it; notwithstanding these two Lines having two Points common, they cannot be different.

PROP. 8. *Two right Lines that intersect, may be conceived in one and the same Plane.* Eucl. 11. Prop. 2. 8

THE Lines DB and EC intersect; having drawn right Lines between AB and AC, there will be a Superficies, which is a Plane: for a right Line may be applied to it, and this Superficies is the Shortest that can be conceived between the Lines AB and AC which are upon the Plane; which being produced, if AD and AE which are parts of the Lines BD and EC, be not found in the same Plane, BD and CE are part upon it, and part above it; which has been shewn to be impossible.

PROP. 9. *EVERY Triangle may be conceiv'd as contain'd in a Plane.* 9

THIS is a Consequence of the preceeding Proposition.

PROP. 10. *THE common Section or meeting of two Planes, is a right Line.* Eucl. 11. Prop. 3. 10

X and Z, intersect. The Ends of their Section are the Points E and F, between which there may be drawn a Line upon Z and one upon X. If these two Lines are not one and the same Line, then there may be drawn between the same two Points more than one right Line; which is impossible. † therefore the Section of two Planes can be no other than that of a right Line. See P. B. Plate 9. fig. 36. or 37.

PROP. 11. *A right Line such as AB raised upon a Plane X, ought to be looked upon as a Perpendicular when its End is equally distant from the Circumference of a Circle; described about its End A, as a Center, and when it is not, it cannot be said to be Perpendicular.* 11

THIS is agreeable to the Nature of a Perpendicular Line, which inclines not more to one Side than to the other.

PROP. 12. *A Line such as AC Perpendicular upon the Plane X, being conceived as carried by a direct and regular Motion according to a right Line such as AB, it will make the Plane Z, which will be Perpendicular in all its parts upon the Plane X.* 12

PROP. 13. *If the Line ED, Perpendicular upon AB, the Section of Z and X, is Perpendicular upon X, all the Plane Z is Perpendicular upon X.* 13

For

* B. 1. n. 16. † B. 1. n. 15.

For the Plane Z may be conceived as described by the direct and uniform Motion of DE upon AB; therefore by the preceeding Proposition, the Plane Z is Perpendicular upon the Plane X.

- 14 THEOREM 1. BETWEEN two Lines either Parallel or not, that are in one and the same Plane, or between a Line and a Point, can be conceiv'd but one and the same Plane, in which is contain'd every right Line that is drawn between any two Lines of that Plane.

For if there could be conceived two Planes, one would be greater or lesser; which cannot be, since every Superficies that is not the least between its Bounds, is not a Plane. * they are then equal; which being so, they are not different: For if it be said that they are placed one upon the other, as they have no thickness, they could make only one Plane.

- 15 THEOR. 2. Two Planes that agree in three Points which are not upon one and the same Line, agree entirely.

1. BETWEEN two of the given Points, can be conceived only one right Line. 2. Between this right Line and the third Point given, by the preceeding Propositions two different Planes cannot be conceived; therefore the Part of these two Planes between these three Points is one and the same, consequently if this part be prolonged, it will be only one and the same Plane; therefore these two Planes are not different.

- 16 COROLLARY. THEREFORE the Position of a Plane depends only upon three Points, which are not upon one and the same Line.

OR which is the same, three Points which are not upon one and the same Line being given, the Plane is given.

- 17 THEOR. 3. IF B, the Top of the Line AB raised upon the Plane X, is equally distant from C, D, E, three Points equally distant from its Foot A, it is Perpendicular upon X.

1. IN the Plane X suppose a Circle at an equal Distance from B, passing thro' C, D, E, which are supposed at equal distances from B. 2. Let us conceive a second Circle, whose Center is A, as passing thro' the three Points C, D, E which by the Hypothesis, are also equally distant from A. These two Circles are the Same. † Therefore AB is Perpendicular upon X; ‡ which was to be proved.

- 18 PROBLEM 1. FROM a Point given above as B, to let fall a Perpendicular upon the Point X. Eucl. 11. Prop. 11. (same Fig.)

DRAW the two right Lines CE and DF at pleasure, and set one foot of the Compasses upon B, with the other take the Points C, D, E, equally distant, thro' which draw a Circle, § to the Center of which draw a Line from B, which will by the preceeding Theorem, be the Perpendicular required.

Theorem

* Sup. n. 2.

† B. 1. n. 88.

‡ Sup. n. 11.

§ B. 1. n. 87.

THEOR. 4. IF a Line is Perpendicular upon the Point of Section of two Lines which are upon a Plane, it is so likewise upon the whole Plane. Eucl. 11. Prop. 4. ¹⁹

FOR having taken Points in these two Lines on each Side the intersecting Point, equally distant therefrom, the Top of the Perpendicular will be at an equal Distance from four Points which are upon the Plane; * therefore this Line will be Perpendicular upon the whole Plane. †

THEOR. 5. THE right Line AB, and all other in the Plane Z drawn thro' A, the Foot of the Perpendicular AC, upon the Plane, is Perpendicular upon AC, or AC is Perpendicular upon all the right Lines of the Plane Z which pass thro' A. ²⁰

FROM A the Foot of the Perpendicular AC, describe a Circle X, and produce the Line BA so that it may cut X in D and B. If C, the Top of the Perpendicular AC, is not equally distant from D and B, then AC will not be Perpendicular upon AB; ‡ But also AC will not be Perpendicular upon the Plane Z; § which is contrary to the Hypothesis. AB therefore meets AC Perpendicularly; as doth likewise every other Line of the Plane Z, which passeth thro' A.

THEOR. 6. IF a right Line AC meets three other right Lines AB, AF, AG Perpendicularly at one and the same Point, the three Lines are in one and the same Plane. Eucl. 11. Prop. 5. (same Figure.) ²¹

LET AB and AF be in the same Plane Z, and that AG is not, but upon another Plane. Let a third Plane Y be conceived to pass thro' AC and AG, and which being prolonged will cut Z; that Line AC will be Perpendicular upon the Plane Z; || and therefore also upon the Section Line of Z and Y, * and it is upon AG by the Supposition, therefore upon AC and at the same Point A, there will be two Perpendiculars, which cannot be. †

PROB. 2. To raise a Perpendicular, upon the Point A in the Plane Z. Eucl. 11. Prop. 12. (same fig.) ²²

FROM the Center A describe a Circle; every Line that is let fall from a Point at an equal Distance from the Circle, is the Perpendicular required.

THEOR. 7. THERE can be but one Perpendicular from the same Point upon the same Plane. Eucl. 11. Prop. 13. ²³

IF there can, let us conceive that the two Lines AE and AC supposed to be Perpendiculars and raised upon the same Point A, and that X be described from A as a Center: The Points E and C the Tops of the two equal Lines AE and AC being different cannot be equally distant from the Circumference of the Circle X; therefore they are not both Perpendicular upon the Plane Z. †

Theorem

* B. 1. n. 42. † Sup. n. 11. ‡ B. 1. n. 40.

§ Sup. n. 11. || Sup. n. 19.

* Sup. n. 20. † B. 1. n. 49. ‡ Sup. n. 10.

THEOR. 8. FROM a Point without a Plane can be drawn but
 24 one Perpendicular upon the Plane.

LET A be the Point given above a Plane from whence there is supposed to be drawn two Perpendiculars, viz. AB and AC; join their lower Ends B and C by the Line BC. The three Points A, B, C, form the Triangle ABC which may be conceived to be in one and the same Plane. * The Angles ABC and ACB formed by the Perpendiculars AB and AC, are right ones; then ABC hath more than two right Angles; which cannot be. †

COROLLARY. IF the Plane X is Perpendicular upon the Plane
 25 Z, and that from the Point A in the Plane X, be let fall a Perpendicular upon Z, it will fall upon MN the Section of Z and X. Eucl. 11. Prop. 38. (same fig.)

IF it be not so, and that a Perpendicular fall from A upon C out of the Line MN; let fall AB Perpendicularly upon MN; therefore as B is in the Section of X and Z. there is upon Y two Perpendiculars AB and AC drawn from one and the same Point A; which has been shewn to be impossible.

THEOR. 9. THE Perpendicular Line is the Shortest that can be
 26 drawn upon a Plane from a Point without the same.

THE given Point is A, the Line AB is Perpendicular, AC is not. It must be proved that AB, is shorter than AC. To do which join C and B by a right Line. The Triangle ABC may be conceived to be in one and the same Plane. ‡ Now AB Perpendicular upon BC, is shorter than the Oblique Line AC; § which was to be proved.

COROLLARY. THEREFORE the distance between a Plane, and
 27 a Point which is out of the Plane, ought to be measured by a Perpendicular.

By reason the Perpendicular is the shortest Line, and that there can be drawn no more than one.

THEOR. 10. Two Lines being Perpendicular upon one and the
 28 same Plane, they may be conceived as being in one and the same Plane.

LET us conceive a third Line BD to be drawn by the bottoms of the two Lines AB and CD which are Perpendicular upon X, upon which third Line let us conceive AB or CD as moving uniformly and always Perpendicularly, which will thereby constitute a Plane, || Q. E. D.

THEOR. 11 AB and CD (same fig.) Perpendiculars upon X, are
 29 parallel; and if one of the two Parallels is Perpendicular upon the Plane X, the other is so likewise. Eucl. 11. Prop. 6 and 8.

1. LET the Line BD be drawn thro' the bottom of AB and CD, which Lines are Perpendicular upon BD; * therefore the three Lines AB, CD, BD may be in one and the same Plane.

* Sup. n. 9. † B. 2. n. 79. ‡ Sup. n. 9.

§ B. 1. n. 53. || Sup. n. 1.

* Sup. n. 20.

Plane. * Therefore AB and CD being Perpendicular upon BD, they are Parallel. † 2. If AB and CD are Parallel, and that AB be Perpendicular, I say that CD is so likewise; for having drawn BD, the Line AB is Perpendicular upon ED; ‡ therefore CD parallel to AB, is also Perpendicular upon BD; § which was to be proved.

THEOR. 12. THE Section AB of the two Planes Z and X, which are Perpendicular upon Y, is a Perpendicular upon the same Plane. ³⁰
Eucl. 11. Prop. 19.

1. THIS Section is a right Line, || and seeing the Planes Z and X are Perpendicular upon Y, the Line AP inſomuch as it is conſidered in Z cannot be conceived as leaning more on one ſide thereof than the other: And by the ſame Reason, inſomuch as it is conſidered in X, it cannot be conceived to incline more to to one ſide of the Plane than to the other; wherefore it may be conceived by the Means of three Points in the Plane Z equally diſtant from P, which are alſo equally diſtant from A; therefore AP is Perpendicular upon the Plane Y. *

THEOR. 13. THE Section of X and Z being Perpendicular upon Y, theſe two Planes and all others whoſe Section is AP, are Perpendicular upon Y. Eucl. 11. Prop. 18. (ſame fig.) ³¹

For all theſe Planes may be conceived as generated by the Motion of AP. †

THEOR. 14. IF three Points in one and the ſame Plane, and not upon one and the ſame Line, are at an equal Diſtance from a ſecond Plane, theſe two Planes are parallel. ³²

THE Position of a Plane depends only upon three Points; ‡ conſequently if three Points of a Plane are equally diſtant from a ſecond Plane, all other Points of theſe two Planes are at an equal diſtance; therefore they are Parallel.

THEOR. 14. THE Planes X and Z being Parallel, if the Line AB is Perpendicular upon X, it is ſo alſo upon Z. And if AB is Perpendicular upon X and upon Z, the two Planes are Parallel. ³³
Eucl. 11. Prop. 14.

If it is pretended that AB is Perpendicular upon X and not ſo upon Z, let us ſuppoſe a Perpendicular at the Point A to be raiſed towards Z ſuch as AC, which is ſhorter than AB. § From C conceive a Perpendicular upon X, which is alſo ſhorter than AC, therefore yet more ſhorter than AB; hence the Point D approaches nearer Z than to A; therefore Z and X not being at an equal diſtance, they are not Parallel; which is contrary to the Hypotheſis. Therefore a Line which is Perpendicular upon one of the Planes, is ſo upon the other. The reſt is eaſy.

THEOR. 16. THE Sections AB and CD of two Parallel Planes cut by a third Plane, are Parallel Lines. Eucl. 11. Prop. 16. ³⁴
The

* Sup. n. 28. † B. 1. n. 68. ‡ Sup. n. 20.
§ B. 1. n. 69. || Sup. n. 10.
¶ Sup. n. 17. † Sup. n. 12. ‡ Sup. n. 16.
§ B. 1. n. 53.

THE Sections AB and CD are right Lines, * which are in the third Plane, from whence being prolonged, they would meet if they were not Parallel; consequently the other two Planes in which they are contain'd being prolonged, would also meet, therefore are not Parallel, contrary to the Supposition; therefore it cannot be said, that AB and CD are not Parallel.

35 THEOR. 17. RIGHT Lines which are Parallel to one and the same right Line, altho' upon different Planes, are Parallel one to another. Eucl. 11. Prop. 9.

LET CE be Parallel to AB, to which DF in another Plane is also Parallel; I say that CE and DF are parallel to each other. From the Point B upon AB raise BC and BD Perpendicularly, crossing CE, and DF, at the Points C and D. By means of the three Points B, C, D, may be conceived a Plane; † AB Perpendicular upon BD, and BC, is Perpendicular upon that Plane, ‡ but CE and DF being supposed parallel to AB, are also Perpendicular upon the said Plane and Parallel to each other; § which was to be proved.

36 THEOR. 18. All parallel Lines in the Plane X, met by other parallel Lines in the Plane Z, form Angles equal one to another. Eucl. 11. Prop. 10. (same fig.)

BD and NO are Parallel to each other upon the Plane X; and BC, and MN upon the Plane Z: It must be proved that the Angle CBD is equal to MNO. To do which, draw DF and CE Parallel to AB. Since that Parallels between Parallels are equal, for they constitute the same Angles, || consequently equal ones; * I say, since CE is Parallel to DF by the preceeding Theorem; therefore $BD=NO$, and $BC=NM$, and $CD=MO$; therefore the Triangles CBD and MNO are equal and similar, therefore the Angle CBD is equal to MNO; Q. E. D.

THEOR. 19. Two Lines AB and AC which meet at the Point A, and are Parallel to two other Lines ED and DF which meet at D, if they are not upon one and the same Plane, the Planes BC and EF, are Parallel. Eucl. 11. Prop. 15.

FROM A let there be drawn a Perpendicular upon the Plane EF which it meets at the Point G, from which draw GH parallel to DE, and GI parallel to DF: Then by Theorem 17, † AB and GH are Parallel, as also AC and GI: Hence AG being by construction, Perpendicular upon GI and upon GH, it is so likewise upon AC and upon AB, ‡ and therefore upon the Plane BAC; § hence the two Planes must needs be Parallel; for if they were not, they (being produced) would meet in some Point or other, from whence might be drawn right Lines to the Points A and G, which therefore are not Parallel, contrary to what has been demonstrated, since AG being Perpendicular upon the

* Sup. n. 10. † Sup. n. 9. ‡ Sup. n. 19.

§ Sup. n. 29. || B. 2. n. 27.

* B. 2. n. 110. † Sup. n. 35. ‡ B. 1. n. 69.

§ Sup. n. 19.

the said Planes, it ought to be so upon all the Lines drawn upon them thro' the Points A and G; * which cannot be if those Lines cou'd meet.

THEOR. 20. Two right Lines being cut by parallel Planes, are divided Proportionally. Eucl. 11. Prop. 17. 38

LET the two right Lines be AB, CD, cut by three parallel Planes X, Y, Z, at the Points A, L, B, D, M, C; I say that they are divided Proportionally, that is, $AL : LB :: CM : MD$. For upon each of the extreme Planes X and Z, the Points of Section are joined by the right Lines AC and BD, and the Diagonal AD being drawn intersects the Plane Y at the Point N, from which are drawn the Lines LN and MN thro' the Points of Section L and M. The Triangle ABD may be conceived in one Plane, † as may also that of ADC in another, if it is not the Same; each of the Planes of these Triangles being cut by parallel Planes, the Section Lines are Parallel, ‡ and the Triangles are also cut Parallel to their Bases, and therefore $AL : LB :: AN : ND :: CM : MD$, § and consequently $AL : LB :: CM : MD$; || which was to be proved. If there had been more Planes and more Lines, it would have been the same thing.

SECT. II.

OF the Composition of Solids according to their Surfaces, and Solidity.

DEFINITION I.

WHEN three or more plane Angles, which are upon different Planes, (or which have not one and the same base) meet at 39 their Vertex, the Angle which they constitute is called a solid Angle.

Two plain Angles only cannot form a Solid Angle; it requires at least three plane Angles, which cross or meet about a Point, like a well cut Diamond.

DEF. 2. SOLIDS bounded by rectilinear Surfaces, or parallel Planes, are called Parallelopipedons. P. B. Plate 7. fig. 20. 40

DEF. 3. SOLIDS contained between Planes all similar and equal, are called Polyedrons. When the Figures which bound them 41 are regular, and all their Angles equal, they are called regular Polyedrons.

DEF. 4. A regular Solid receives a different and particular Name, according to the Number of its planes or Surfaces. 4

U

1. When

* Sup. n. 20. † Sup. n. 9. ‡ Sup. n. 134.

§ B. 4. n. 16. || B. 3. n. 53.

1. WHEN it is bounded by four equal and Equilateral Triangles, it is called a, Tetraedron. See P. B. Plate 1. fig. 3.

2. HEXAEDRON, or Cube, when it is bounded by six equal Squares, See P. B. Plate 1. fig. 1.

3. OCTOEDRON, when bounded by eight equal and Equilateral Triangles. P. B. Plate 1. fig. 4.

4. DODECAEDRON, when it's bounded by twelve equal and equilateral Pentagons. P. B. Plate 1. fig. 6.

5. ICOSAEDRON, when bounded by 20 equal and equilateral Triangles. P. B. Plate 1. fig. 5.

DEF. 5. A Line whose Top is unmoveable, and its bottom made
43 to move round a Figure. 1. If the Figure be a Polygon, the Line will describe a Solid, which is called a Pyramid, such as ABCD. 2. If the Figure about which the bottom of the Line moves is a Circle, the Solid generated thereby is a Cone, as X. P. B. Plate 7. fig. 28. and Plate 8. fig. 31.

In these Figures the Line drawn from the Vertex to the Center of the Base, is called its Axis, and the Solid is called either Right or Oblique, according as its Axis makes either a right or an oblique Angle with the Base.

3. If the Top of the Line, which describes the Solid is not unmoveable, but that whilst its bottom moves round the Base, its top moves round a Similar, equal Figure, and alike posited, such as ABCDEF, it is a Prism. P. B. Plate 7. fig. 19, 21, 22, &c.

4. If the Base is a Circle, it will be a Cylinder, such as ABXZ.

DEF. 6. If a Circle revolves about its Diameter, it describes a
44 Sphere, the Diameter of which is called the Axis thereof; the Center of the Circle, whose revolution makes the Sphere, is the Center of the Sphere; the Lines drawn from the Center to the Circumference are the Radii; those which pass thro' the Center and terminate at the Circumference are Diameters.

DEF. 7. A regular Polygon revolving about a right Line
45 which passeth thro' its Center, describes what is called a Spheroid, that is, a different Sort or Species of a Sphere, such as is X and Z.

DEF. 8. A Solid A is said to be Circumscribed about another
46 Solid B which it contains, if it is the least of all similar Solids capable of containing B; or rather, if B is the greatest of all similar Solids which A is capable of containing.

DEF. 9. A Solid B is said to be inscribed in another Solid A in
47 which it is contained, if it is the greatest of all similar Solids that can be contained in A; or which is the Same, if A is the least of all similar Solids that can contain it.

DEF. 10. A regular Body is that which is bounded by regular,
48 Similar, and equal Figures, whose solid Angles are also all equal.

It will be hereafter demonstrated, that there are only five regular Bodies.

THEOREM 1. *If a solid Angle is comprized under three plane Angles, any two of these plane Angles taken together, are greater than the Third.* Eucl. 11. Prop. 20. ⁴⁹

THE three plane Angles BAD, BAC, CAD make a Solid Angle; it must be proved that two of these plane Angles taken together are greater than the Third; as for Example BAD + BAC + CAD, there is no lesser Plane than BAD, between the Lines AB and AD; * therefore the rectilinear Plane BAD is lesser than the Curvilinear one ABCD. In the same manner it is demonstrable, that the plane Angle BAC is lesser than BAD and CAD together.

COROLLARY. *THE Sides of three plane Angles, which make a solid Angle, being taken equal, the Bases of these three plane Angles, make a Triangle.* Eucl. 11. Prop. 22. (same fig.) ⁵⁰

LET the three Lines AB, AC, AD, Sides of three plane Angles, which make the solid Angle A, be taken equal. It must be demonstrated that the three Lines BC, CD, DB. Bases of the three plane Angles BAC, CAD, DAB, make a Triangle. To do which it is enough that two of these Lines taken together, are longer than the Third. † Now this is so, for the two Angles of which these Lines are the Bases, are greater than the Angle which is upon the Third; and consequently this third Line is shorter. ‡

THEOR. 2. *ALL the plain Angles which comprize a solid Angle, are together less than four right Angles.* Eucl. 11. Prop. 21. ⁵¹

1. LET us begin with the solid Angle A, composed of three Planes: by the preceeding Theorem the Angles BCA + DCA are greater than the Angle BCD, and likewise ADC + ADB greater than CDB; as also ABD + ABC greater than DBC. Hence the six Angles at the Bases of the three Triangles which form the solid Angle A, are greater than the three Angles of the Triangle BCD, that is, greater than two right Angles. § Now all the Angles of the three Triangles which form the said solid Angle A, are together equal to six right ones. || If then from that Sum be taken more than two right ones, the Content of the six Angles at the Base, there will remain less than four right Angles for the Content of the solid Angle A; which was to be proved. 2. By the same Method it is shewn that the solid Angles formed by four Planes, are less than four right ones. 3. If X is a solid Angle comprehended under five Triangles, whose angular Point ought to be conceived as above the Plane, the preceeding Theorem will also prove that the Angle DBC is lesser than the two Angles DEA and CBA, and the Angle BCE is lesser than the Angles ACB and ACE taken together, the same of the others. But also all the Angles of the Polygon BCEFD,

* Sup. n. 3. † B. 2. n. 67. ‡ B. 2. n. 104.

§ B. 2. n. 75. || B. 2. n. 75.

U 2

the

the Base of the solid Angle, are equal to six right ones; * hence all the Angles at the Base of the five Triangles which make the solid Angle X, are greater than six right ones, seeing they are greater than the Angles of the Polygon, as was shewn before. All the Angles of the five Triangles which make the solid gle are equal to ten right ones; therefore since those of their Bases are more than six right ones, those of their Vertices are less than four right ones. 2. E. D.

THIS Theorem may be demonstrated thus.

1. LET us conceive A to be a Point in a Plane, and the Vertex of several Triangles whose Bases are the Sides of X. All the Angles about A make only four right Angles. 2. If the Point A be raised up, then the vertical Angles of the Triangles will become less, having the same Bases and longer Sides; † this being so, all the Angles will together be less than four right angles.

52. PROBLEM 1. HAVING three plane Triangles, two of which taken together are greater than the Third, but which all together are less than four right Angles, to make therewith a solid Angle. Eucl. 11. Prop. 23.

LET the three plane Triangles be A, B, C, 'tis required only to join them in one and the same Point, so that the containing Sides may coincide together, that is, that AE unite with BE, BF with CF, and CG with AD; this will form the solid Angle required, according to the Definition thereof. ‡

53. PROB. 2. To make a solid Angle upon a given right Line, and at a Point given in that Line, equal to a solid Angle given. Eucl. 11. Prop. 26.

LET the given Line be AD, and A a given Point in that Line, required to make at that Point, a solid Angle, equal to a solid Angle given. Let the three Planes of the given Angle be DAE, EBF, FCG; having made three other Planes equal thereto, and joined them together at the Point A in the Manner as was explained in the preceeding Problem, it is evident that they constitute the solid Angle required.

54. THEOR. 3. If there are two equal plane Angles, thro' the Vertices whereof are drawn two Lines, raised upwards, and containing equal Angles with the Lines first given, each to each, and from a Point taken in each of those elevated Lines, be drawn lines Perpendicular to the Planes, which contain the Angles first given, and from the Points where those Perpendiculars fall, be drawn right Lines to the Vertices of the Angles first given, the Angles which those Lines make with those elevated Lines, are equal one to another. Eucl. 11. Prop. 35.

IN Euclid this Proposition is used for the Demonstration of other following ones, but as the Method of this Treatise has no need of it, I therefore pass it without taking any farther Notice thereof.

Theorem

* B. 2. n. 123.

† B. 2. n. 99.

‡ Sup. n. 39.

THEOR. 4. THERE cannot be more than five different regular Bodies.

55

THAT is five different Solids comprized under plane regular Figures all equal and similar, and whose Angles are all equal. In a Sphere, the whole is equal; but it is different in a Solid comprized under Planes. It has been proved that all the plane Angles, which form a solid Angle, are less than four right Angles. Let us now see which are the Plane, Similar, Equal, regular Figures that can make a solid Angle. 1. Three equal and equilateral Triangles may make a solid Angle, for the three Angles of the Triangles which contain a solid Angle, make only three times sixty Degrees, each Angle of an equilateral Triangle being sixty: Three of these Triangles joined together may therefore make a solid Angle, such as are those of the Tetraedron. 2. Four of these Triangles may also make a solid Angle; for the four Angles which they contain are only four times Sixty; which is less than four right ones. Four of these Triangles make the Angle of the Octaedron. 3. Five of these Triangles may also compose a solid Angle, for the Angles of the Planes which they comprize, make only five times sixty Degrees; which is less than four right Angles. The Angle of the Icosaedron is made by five of these Triangles.

Six equilateral Triangles cannot make a solid Angle; for the six plane Angles which they contain, make four right Angles, 6 times 60=360 the Sum of four right Angles; therefore these six Triangles make a plane Angle, and not a Solid one. 4. Three Angles of a Square may make a solid Angle; for their Sum is less than four right Angles. The Angle of the Cube is composed of three Angles of the Square. No other Solid can be conceived as composed of Squares for four cannot make a solid Angle; much less, either five or six. 5. Three Pentagons form a solid Angle; for their Angles make only 324 Degrees, which is less than four right Angles. Each angle of the Dodecaedron is contained under three Pentagons, more than three Pentagons cannot make a solid Angle; for four make 432 Degrees, which is above the Sum of four right Angles. Three Hexagons cannot constitute a solid Angle; for each being 120 Degrees, the three make 360, which is the Sum of four right Angles: Therefore they cannot make a solid Angle. † The more Sides a Polygon has the greater is the Angle thereof; therefore if three Hexagon's cannot make a solid Angle, there is no possibility that three Heptagon's should do it; hence there is only the *Equilateral Triangle*, *Square* and *Pentagon* that can do it: But a solid Angle may be made of three, four, and five equilateral Triangles, there cannot therefore be any more than five different regular Bodies.

Evident

† Sup. n. 5.

Evident Propositions.

PROPOSITION I.

56 **A** Figure is greater than that about which it is circumscribed, and lesser than that in which it is inscribed.

57 **PROP. 2.** Two Prisms having the same Height, that which hath the least Base, (and which may consequently be contained in the other) is the least.

For it is evident that it is contained therein. Now that which contains, is greater than that which is contained.

58 **PROP. 3.** Of two Prisms circumscribed about a Cylinder, that which is most like the Cylinder, has the most Sides.

THE Base of the proposed Cylinder is X; let that of the Prism which hath fewest Sides, and which is circumscribed about the Cylinder, be called Y, and Z that of another Prism having more Sides, and circumscribed about the same Cylinder. The Polygon Z is lesser than the Polygon Y. * The Prisms, whose Bases are Polygons are of the same Height, being Circumscribed about one and the same Cylinder, therefore that which is upon Z, and which hath therefore most Sides, is lesser than that which hath fewer, and whose Base is Y; † it therefore approaches nearest to the Cylinder; *Q. E. D.*

59 **PROP. 4.** THEREFORE a Prism whose Sides are infinite, is so near like a Cylinder, that there is no perceivable difference between them, the Cylinder may therefore be supposed to be a Prism having an infinite Number of Sides.

60 **PROP. 5.** Of two Prisms inscribed in a Cylinder, that which approaches nearest to a Cylinder, hath most Sides.

THE Base of the proposed Cylinder is X, the two inscribed Polygons Z and Y are the Bases of two Prisms inscribed in the Cylinder whose Base is X. The Polygon Y which hath most Sides is greatest, and approaches nearer to a Circle than the Polygon Z. ‡ The two Prisms, whose Bases are Z and Y, are of the same Height, being inscribed in one and the same Cylinder; therefore the Prism which is upon Y is greater than that which is upon Z. § It therefore approaches nearest to a Cylinder; *Q. E. D.*

61 **PROP. 6.** Two Pyramids having the same Height, that is the greatest which hath the longest Base.

For it is evident that if one be conceived as put upon the other, that which hath the greatest Base will contain that which hath a lesser.

62 **PROP. 7.** Of two Pyramids circumscribed about a Cone, that which hath most Sides most resembles a Cone.

Prop.

* B. 2. n. 149.

† Sup. n. 57.

‡ B. 2. n. 151.

§ Sup. n. 56.

PROP. 8. OF two Pyramids inscribed in a Cone, that which hath most Sides approaches the nearest to a Cone. 63

PROP. 9. A Cone may therefore be supposed to be a Pyramid of infinite Sides. 64

PROP. 10. THE more sides a Polygon hath, the nearer the Spheroid which it forms approaches to its inscribed Sphere. 65

FOR the more Sides a Polygon (which may be called the Generator of a Spheroid) hath, the nearer it approaches to a Circle; therefore the Spheroid which it describes by its revolution approaches nearer its inscribed Sphere. Q. E. D.

PROP. 11. THEREFORE a Sphere may be taken for a Spheroid formed by a Polygon of an infinite Number of Sides. 66

S E C T. III.

OF the Surfaces of Solids.

THEOREM I.

THE Surface of a right-angled Prism is equal to a Parallelogram having the same Height, and its Base equal to the Circumference of the Prism. 67

THE Superficies of a right angled Prism are Parallelograms having all the same Height, whose Bases taken together make the Circumference of the Prism: It is therefore equal to a Parallelogram having the same Height as the Prism, and whose Base is equal to its Circumference; which is evident. The Surfaces of the Bases of the Prism are excluded.

COROLLARY 1. THEREFORE seeing that a right Cylinder may be taken for a Prism of infinite Sides, * its Surface is equal to a Parallelogram of the same Height and its Base equal to the Circumference of the Circle, which is the Base of the Cylinder. 68

COROL. 2. THEREFORE all that has been demonstrated concerning the Ratio which Parallelograms have one to another, agrees to Cylindrical Surfaces. 69

1. THE Ratio which the Surfaces of two Cylinders bear one to another is compounded of their Heights, and of the Circumference of their Bases. 2. In two Cylinders, if it be as Height is to Height so is Base to Base, that is, if the Ratio of their Surfaces is compounded of two equal Ratio's, the Ratio is duple. † 3. If two Cylinders have either their Heights or their Bases equal, their Surfaces are one to another as their parts which are unequal. ‡

Theorem

* Sup. n. 59.

† B. 4. n. 71.

‡ B. 4. n. 75.

THEOR. 2. If a Prism hath a Polygon for its Base, and the Apotome thereof for its Height, its Surface is double that of the Polygon which is its Base.

LET Z be a Prism, whose Base is the Polygon X. The Apotome of X is AG, that is, a Perpendicular drawn from its Center A upon one of its Sides BC; It must be demonstrated that if FG, the Height of Z is equal to AG the Apotome of X, the Surface of Z is double that of X. Having drawn Lines from each Angle of the Polygon X to the Center A, they make as many Triangles as Z hath Sides or Faces, which Faces are Parallelograms. Now these Parallelograms as BCED, and the Triangles as ABC, have the same Height and Base. Therefore the Parallelograms are double the Triangles, * and consequently the Surface of Z compounded of the Parallelograms is double that of X, equal to all the Triangles.

71 COROLLARY 1. THEREFORE if the Height of a Cylinder (which may be esteemed as a Prism) is equal to the Radius of the Circle which is its Base, its Surface is double that of the Circle.

72 COROL. 2. THEREFORE if the Height of a Cylinder be twice the Radius, or once the Diameter of the Circle which is its Base, its Surface is Quadruple that of the Circle.

73 THEOR. 3. THE Surface of the Circumference of a Cylinder is equal to that of a Circle, whose Radius is a mean Proportional between the Height of the Cylinder and the Diameter of the Circle which is its Base. Archimedes, 1. Prop. 16.

X is a Cylinder, whose Height is AB, the Circuit of its Base which is the Circle Z, is BD, whose Circumference is FG or BD, and BC, the Double of the Circuit. The Surface of X is equal to the Parallelogram ABD or to the Triangle ABC, as is that of the Circle Z to the Triangle EFG. † I suppose KH, the Radius of the Circle Y, to be a mean Proportional between the Height of X, and 2 EF the Diameter of Z. Let KL be the Circuit of Y, hence its Surface is equal to the Triangle HKL; it is therefore required only to prove that the Triangle HKL is equal to the Triangle ABC: For all Circles have same the Ratio between the Radius and Circumferences.

By the Hypothesis $\therefore AB. HK. 2EF.$ Now $HK. KL::2EF. 2FG$ or BC; and Alternately $HK. 2EF::KL. BC$; then seeing that $AB. HK::HK. 2EF::KL. BC$. Hence $AB. HK::KL. BC$, † wherefore $AB \times BC = HK \times KL$. § Now the Triangles ABC and HKL are the Halves of the Rectangles; therefore they are equal; which was to be proved.

74 THEOR. 4. THE Surface of a Pyramid is equal to a Triangle, whose Height is equal to the Height of each of its Sides, and whose Base is equal to the Circuit of the Base of the Pyramid, or to a Parallelo-

* B. 2. n. 133. † B. 2. n. 154. ‡ B. 3. n. 53.

§ B. 3. n. 56.

Parallelogram having the same Height, and but half the Base.
Archimedes 1. Prop. 10 and 11.

EACH Side of the Pyramid is a Triangle; these Sides being equal, the Triangles are equal one to another, and to a right Angled Triangle having the same Height, and its Base equal to all the Bases of these Triangles taken together. * Now a Triangle is equal to a Parallelogram having the same Height, and half the Base, † which was to be proved.

COROLLARY 1. THEREFORE seeing that Cones may be considered as Pyramids, the Surface of a Cone is equal to a right angled Triangle having the same Height as the Side of the Cone, and its Base equal to the Circuit of the Base of the Cone, or to a right angled Parallelogram having the same Height, and whose Base is equal to half the Base of the Cone. 75

THE superficial Height of a Cone and of a Pyramid is the shortest right Line that can be imagined possible to be drawn upon the Surface, from the Vertex down to its Base.

COROLL. 2. THEREFORE all that has been demonstrated of 76 the Ratios and proportions between several similar Rectangles, agrees to the Superficies of Cones.

1. THE Surfaces of Cones are one to another in a Ratio compounded of their Height and Base. 2. If Height is to Height as Base to Base, the Surfaces are in duple Ratio. 3. If two Cones have equal Heights or equal Bases, their Surfaces are one to another as their parts which are unequal. 4. Therefore seeing that the Circumferences of Circles are one to another as their Diameters, the Surfaces of two Cones having the same Height, are one to another as the Diameters of their Bases. 5. A Rectangle being given, there may be found one or more similar ones having therewith, any Ratio you please; also a Cone being given there may be found one or more other similar Cones bearing to the given one any Ratio required.

THEOR. 5. THE Surface of a Cone X is to that of the Circle 77 BCD, which is its Base, as AB the Height of its Surface is to the Radius of the Circle. Archimedes 1. Prop. 18.

THE Surface of the Circle is equal to the right angled Triangle Z whose side D is equal to the Radius BC, and the side E to the Circumference of the Circle. † The Surface of the Cone X is equal to the rectangled Triangle Y, whose side F is equal to AB, and the side G to the Circuit of its Base. § G and E being each equal to the Circumference of the Circle, which is the Base of the Cone, they are equal one to another; wherefore the Surfaces of these two right angled Triangles Y and Z, which have equal Bases, viz. G and E, are one to another as F to D; || but $AB = F$, and $BC = D$: therefore X the Surface of the Cone, is to that of the Circle which is its Base, as AB the
X height

* B. 2. n. 143.

† B. 2. n. 133.

‡ B. 2. n. 154.

§ Sup. n. 75.

|| B. 4. n. 75.

height of its Surface is to BC the Radius of the Circle which is its Base; *which was to be proved.*

78 THEOR. 6. THE Surface of a Circle whose Radius is a mean Proportional between the Height of the Cone, and the Radius of its Base, is equal to the Cone's Surface. Archimedes 1. Prop. 17.

LET X be a Cone, the height of whose Surface is AB, and the Circuit of its Base BD; hence its Surface is equal to the right angled Triangle ABD. The Line BC is the Radius of its Base. Suppose EF to be a mean Proportional between AB and BC, and also the Radius of a Circle whose circuit is FG; hence the Surface of the Circle is equal to the Triangle EFG: Consequently 'tis required only to prove that $ABD = EFG$. By the Hypothesis $AB : EF :: EF : BC$, and seeing that the Circumferences of Circles are one to another as their Diameters, $EF : BC :: FG : BD$. Hence $AB : EF :: EF : BC :: FG : BD$. then $AB \cdot EF :: FG \cdot BD$; therefore $AB \times BD = EF \times FG$. * Now ABD is the half of $AB \times BD$, † as also EFG is the half of $EF \times FG$. Those Triangles are therefore equal; *which was to be proved.*

79 THEOR. 7. If the Height and Base of a Prism are equal to the Height and Base of a right angled Pyramid, the Surface of the Prism is double that of the Pyramid.

EACH side of the Prism is a Parallelogram, and each side of the Pyramid is a Triangle. In the Case proposed these Parallelograms and Triangles are of the same Height, and the same Base; the Parallelograms are therefore double the Triangles; † therefore the Surface of the Prism is double that of the Pyramid; *which was to be proved.*

30 COROLLARY. THEREFORE seeing Cones may be esteemed as Pyramids, and Cylinders as Prisms, the Surface of the Cylinder may be said to be double that of the Cone having the same Height and Base.

TAKE notice, that the real Height of a Pyramid, must not be confounded with that of the plane Surfaces which form it; the First is a Line drawn from the top of the Pyramid perpendicularly upon the Plane which is its Base; and the Second a different Line drawn also perpendicularly from the Vertex of the Pyramid, but upon one of the Sides of the Polygon which is its Base, which second Line is longer than the First. It is the same of Cones, whose real Heights or Axes are different from those of their sides.

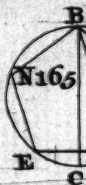
81 LEMMA 1. THE Surface of BCDE, the Frustum of the Cone ADE, is equal to that of the Trapezia GHLK.

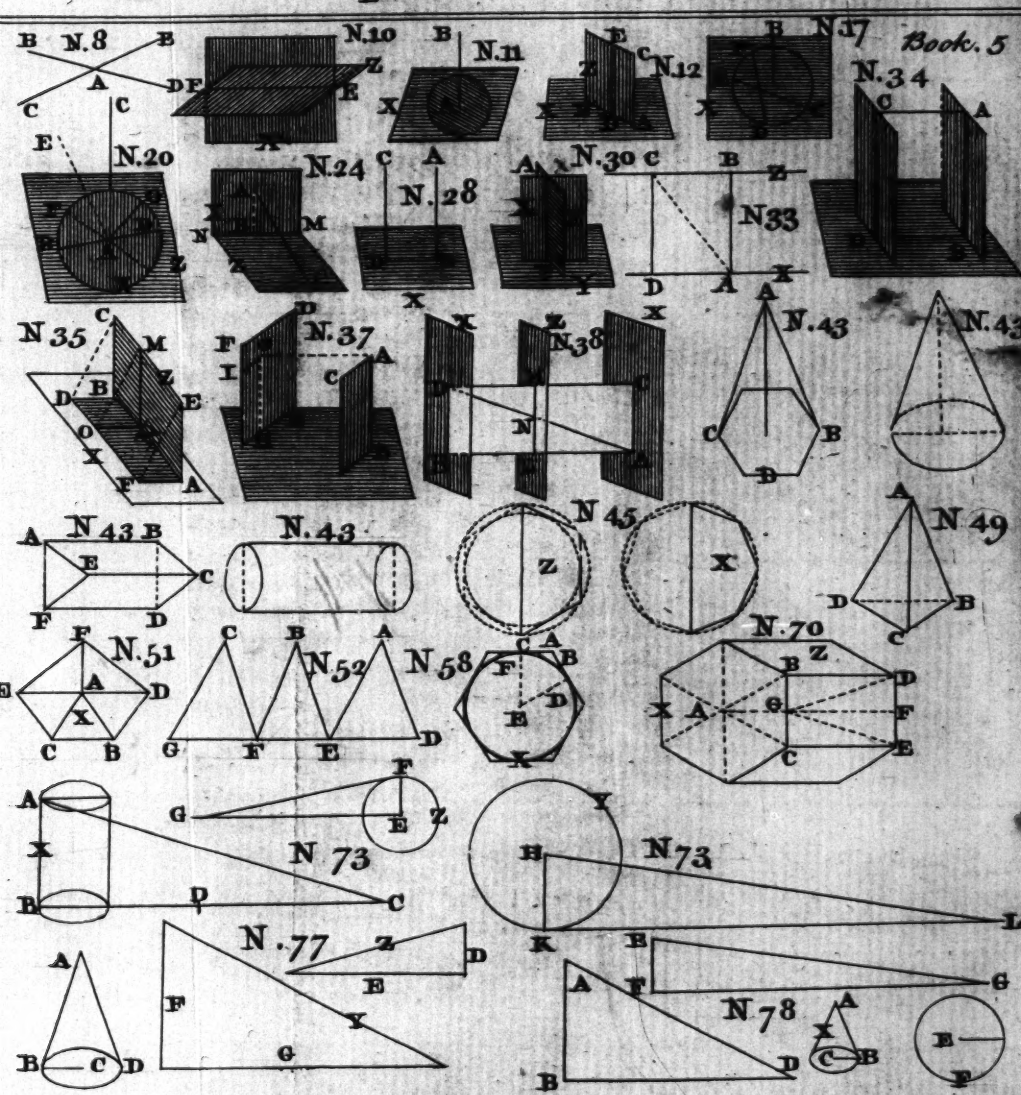
THE Surface of the Cone AED is equal to the right angled Triangle FGH of which $FG = AD$, and GH equal to the Circumference of the Circle DE, § and that of the Cone ABC to the right angled Triangle FKL of which $FK = AB$, and KL

to

* B. 3. n. 56. † B. 2. n. 133. ‡ B. 2. n. 133.
§ Sup. n. 74.

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to the Circumference of the Circle CB: Then taking away FKL from FGH, the Remainder GHLK is equal to the Frustum BCDE, which is evident.

LEMMA 2. HC and KL are the Radii of the two Circles of the two Bases of the Frustum of a Cone. KC is divided into two equal parts at M, and MN is parallel to KL and HC: I say that the Surface of the Frustum is equal to the Rectangle made of KC, the Height of that Surface, and of the Circumference of a Circle, whose Radius is MN.

FOR the Figure CKLH=OCKP, because of the equality of the two Triangles HON and NPL, added or subtracted, but the Figure CKLH is to that which has been proved equal to the Frustum of the Cone, in a Ratio of the Radius of the Circle to its Circumference *. And OCKP is also in the same Ratio to the Rectangle made of KC by the Circumference of the Circle whose Radius is MN; therefore the Rectangle is equal to the Frustum; † 2. E. D.

LEMMA 3. If the Angles of a Spheroid as X, are joined by Planes perpendicular to its Axis, which divide it into several parts, the Parts are either Cones as C, or Frustums of a Cone as B, or Cylinders as A. Which is evident.

THEOR. 8. EACH Surface of the Portions of a Spheroid is equal to the Rectangle made of its corresponding part of the Axis, and Circumference of the Circle or Sphere inscribed in the Spheroid.

As to the Part A, it is very easy, seeing that 'tis a Cylinder whose Surface is equal to the Rectangle of EF by the Circumference of a Circle whose Diameter is EII, ‡ which is equal to the Diameter of the Circle or Sphere inscribed in the Spheroid X; which was to be proved.

Concerning the Part B, it must be demonstrated that the Surface of the Part B (which is the Frustum of a Cone) is equal to a Rectangle made of KL the corresponding Part of the Spheroid's Axis, and of the Circumference of the Circle inscribed in the said Spheroid, whose Diameter is CN. Divide the Height KL into two equal Parts, by drawing CM parallel to FK and to EL: Also draw GE parallel to KL which is equal thereto. The Triangles EFG and ACD are right Angled; therefore GFE and GEF make a right Angle, the Angle GFE being therefore equal to FCD, § the Angle FCD subtracted from the right Angle FCA, the Remainder DCA is equal to GEF, hence the two Triangles ACD and EFG are equiangular: Then GE or KL: EF::CD. CA, || wherefore KL is to EF as the double of CD, which is GM, is to the double of AC, which is CN. Let CM be the Diameter of a Circle whose Circumference is Y; and CN that of a Circle whose Circumference is Z. Then CM. CN::Y. Z. Then

X 2

KL.

* B. 4. n. 75. † B. 3. n. 52. ‡ Sup. n. 68.

§ B. 2. n. 25. † B. 4. n. 10.

KL. EF::Y. Z. Then $KL \times Z = EF \times Y$. * Now the Surface of the Frustum of the Cone B, is equal to $EF \times Y$. † Therefore that Surface equal to $EF \times Y$, is equal to a Rectangle made of KL by Z, the Circumference of a Circle whose Diameter is CN; which was to be proved.

For the Part C. From O the Middle of BD the Side of the Cone C, draw to A, the Center of the Circle inscribed in the Spheroid, the Line AO; and thro' D a parallel to AO; hence as BG is the half of BD, the half of DF is AO; therefore DF is the Diameter of the Circle whose Radius is AO. The Surface of the Cone C is equal to a right angled Triangle whose height is BD, and the Base a Circle whose Diameter is DG, ‡ or to a right angled Parallelogram, having the same height BD, and whose Base is half that of a Circle whose Diameter is DG, § or which is the same, whose Base is equal to a Circle having for its Diameter DE. Let the Circumference of the Circle be called Y, and Z that of the Circle whose Diameter is DF: It must be proved that $BD \times Y = BE \times Z$. The two Triangles DEF and DEB are Similar: || therefore BD. BE::DF. DE. Now DF. DE::Z. Y: * Then BD. BE::Z. Y: † therefore $BD \times Y = BE \times Z$; which was to be proved.

85 THEOR. 9. THE Surface of a Spheroid is equal to the Rectangle made of its Axis, by the Circumference of the Circle of the inscribed Sphere.

By the preceeding Theorem, seeing that the Surface of each part of the Spheroid is equal to the Rectangle, made of each corresponding part of its Axis, and the Circumference of the Circle of the inscribed Sphere, all the whole Surface is equal to the Rectangle of the whole Axis by the Circumference of the Circle of the Sphere inscribed, since that the whole and its parts when multiplied by one and the same Quantity give equal products. ‡

86 THEOR. 10. THE Surface of a Sphere is equal to the Rectangle of its Axis, and the Circumference of a Circle having the same Diameter as that Sphere.

THE Sphere may be esteemed as a Spheroid formed by a regular Polygon having infinite Sides, § whose Diameter may consequently be taken for the Axis of the Sphere. Hence its Surface is, (by the preceeding Theorem,) equal to the Rectangle made of its Axis by the Circumference of the Circle or Polygon, by the Revolution of which it was formed, the Diameter whereof is consequently the same as that of the Sphere; 2 E. D.

Theorem

* B. 3. n. 56.	† Sup. n. 82.	‡ Sup. n. 75.
§ B. 2. n. 133.	B. 4. n. 27.	
* B. 4. n. 42.	† B. 3. n. 56.	‡ B. 3. n. 17.
§ Sup. n. 66.		

THEOR. 11. THE Surface of a Sphere is equal to the Super-87
ficies of its circumscribing Cylinder, which hath consequently the
same Height or Axis.

THE Surface of the Sphere AMNC is equal to a Rect-
angle contained under its Axis, and the Circumference of a
Circle having the same diameter MN. * Now the Surface of the
circumscribing Cylinder, (whose Sides DP and EQ are equal to
AC the Axis of the Sphere) is equal to the same rectangle; for it
is equal to the rectangle made of PD by the circumference of the
Circle of its Base having for its diameter PQ equal to MN; see-
ing that the Diameter of a Sphere inscribed in a Cylinder ought
to be equal to that of the Cylinder's base, according to the Idea
which has been given of inscribed Figures. †

THEOR. 12. If a Sphere inscribed in a Cylinder be divided by 88
Planes perpendicular to its Axis, the Surface of each Part of the
Sphere is equal to that of its corresponding Part of the Cylinder.

AC the Axis of the Sphere is the Height of the circumscrib-
ing Cylinder, therefore the Cylinder touches the Sphere by its
two Bases. Divide the Axis AC by perpendicular Planes, which
will also divide the Cylinder. I say that the Surface of the part
MHIN is equal to that of the Part MGFN of the Cylinder, as
is also the surface of HAI to that of EFGD. For this Sphere
may be considered as a Spheroid, ‡ and the Parts MHIN and
HAI as portions of Spheroids. Therefore the Surface of MHIN is
equal to the Rectangle of BO by the Circumference of a Circle
whose Diameter is MN § which Rectangle is equal to the Sur-
face FGMN; || so likewise the Surface of HAI is equal to the
Rectangle contained under AB and the Circumference of a Circle
whose Diameter is GF, * which the Surface of the Part DEFG is
equal to. †

PROBLEM 1. To divide a Sphere by a Plane, so that the Sur-89
faces of the Portions of that Section may be in a given Ratio. Ar-
chimedes 2. Prop. 4 and 5.

THE Sphere must be inscribed in a Cylinder, then the Sides of
the Cylinder being divided according to the given Ratio, and Lines
or Parallel Planes drawn thro' the Points of that Section they will
divide the Sphere according to the given Ratio; for the Surfaces
of the Portions of the Sphere contain'd between the Parallels, are
equal to those of their corresponding Cylindrical ones, as has
been proved.

THEOR. 13. THE Surface of a Sphere is equal to that of four 90
times its greatest Circle. Archimedes 1. Prop. 37.

LET us conceive a Sphere as inscribed in a Cylinder, whose
Base is consequently equal to the greatest Circle of the Sphere,
and its Height is the Diameter thereof; consequently the Cylin-
drical Superficies is equal to four times the Surface of the Circle.

NOW

* Sup. n. 86.

† Sup. n. 46.

‡ Sup. n. 66.

§ Sup. n. 84.

|| Sup. n. 68.

* Sup. n. 84

† Sup. n. 68.

† Now the Surface of the Sphere is equal to this Superficies: § therefore it is equal to four times its greatest Circle.

- 91 THEOR. 14. THE Surface of a Sphere is equal to that of a Circle, whose Radius is equal to the Diameter of the greatest Circle of the Sphere.

Let X be a Sphere and Z a Circle, whose Radius is equal to the Diameter of the greatest Circle of the Sphere X. It must be proved that the Surface of X is equal to that of the Circle Z. Having supposed the Diameter of the greatest Circle to be 1; then, according to the Hypothesis, the Diameter of Z is 2. Now the Surfaces of Circles being one to another as the Squares of their Diameters, || seeing that the square of 1 is 1, and that that of 2 is 4, according to the Hypothesis, the Surface of Z is quadruple that of the Circle of the Sphere X. Now the Surface of that Sphere is quadruple that of its greatest Circle by the preceding Theorem, it is therefore equal to that of Z; which was to be proved.

- 92 THEOR. 15. THE whole Surface of a Cylinder, that is, all its circumference from its two Bases, is to that of its inscribed Sphere, in a Sesquialter Ratio. Archimedes 1. Prop. 39.

1. THE Circumference of the Cylinder only is equal to the Surface of the Sphere. * Each of the Cylinder's Bases is the greatest Circle of the Sphere, which is the fourth Part of its Surface; † hence the two Bases of the Cylinder are the half of the Sphere's Surface. Wherefore the whole Surface of the Cylinder is equal, 1. to once the whole Surface of the Sphere. 2. To half that Surface; therefore the Ratio is Sesquialter, that is, as 3 to 2.

- 93 THEOR. 16. A Sphere being divided by a Plane, into two Parts, the Surfaces thereof are one to another as the Surfaces of those Circles whose Radii are the Chords of half those Portions, which said Portions are equal to those Circles.

LET X be a Sphere divided by the plane BC; it must be proved that the Surfaces of those two Portions are one to another as the Surfaces of those Circles, whose Radii are AB and BD the Chords of half those two Portions, and that they are equal thereto. 1. Having inscribed the Sphere X in a Cylinder having the same Height, the Surface of the Portion ABC is equal to the circumference of its corresponding Part of the Cylinder, as is that of BCD to the other Part of the Cylinder, † the Circumferences of these two parts are one to another as AE to ED. § Now the Squares of AB and BD are also as AE to ED: ¶ Therefore the Surfaces of the Portions of this Sphere are one to another as those two Squares, or as the Circles whose Radii they are. || The Square of AD is equal

† Sup. n. 72.

§ Sup. n. 87.

|| B. 4. n. 93.

* Sup. n. 87.

† Sup. n. 90.

† Sup. n. 88.

§ Sup. n. 69.

¶ B. 4. n. 80.

|| B. 4. n. 93.

equal to the Squares of AB and BD; * therefore the Surface of the Circle whose Radius is AD, is equal to the Surfaces of the two Circles, whose Radii are AB and BD. † Now the Surface of the Circle whose Radius is AD, is equal to that of the Sphere X: ‡ therefore this Surface is equal to that of the two Circles whose Radii are AB and BD. But it was said that they are also one to another as these Portions; therefore they are each equal to their corresponding one, seeing the same whole is divided in the same Proportion.

THEOR. 17. *THE Surface of a Sphere is double that of the Circumference of the inscribed Cylinder, having its Height equal to 94 the Diameter of its Base.*

THE Surface of the Sphere Z is Quadruple that of the Circle, whose Diameter is A. * The Surface of the Circle whose Diameter is A, is equal to the Surface of the Circles whose Diameters are C and B, seeing that these Surfaces are as the Squares of their Diameters, and that $AA = CC + BB$. † Now $C = B$: Therefore the Surface of the Circle whose Diameter is A, is equal to twice that of the Circle whose Diameter is C; and consequently since that the Surface of A is the fourth Part of the Surface of the Sphere Z, that of the Circle whose Diameter is C is the eighth Part of that of the Sphere Z. Now the Surface of the same Circle, whose Diameter is C, is the fourth Part of the inscribed Cylinder's Surface: ‡ Therefore this Circumference is half the Surface of the Sphere Z.

S E C T. IV.

OF the Solidity of Solids.

EVIDENT Propositions concerning their Solidity.

PROPOSITION 1,

EVERY Parallelepipedon may be conceived as described by its Plane or Base, moving always parallel to itself. 95

PROP. 2. IF the Direction of the Plane whose motion describes the Parallelepipedon, be according to its perpendicular Height, it is right and its Angles are right: If it is according to an Oblique Line, it is Oblique, and then its Angles are not right ones. 96

PROP.

- * B. 4. n. 78. † B. 4. n. 93. ‡ Sup. n. 91.
 * Sup. n. 70. † B. 4. n. 78. ‡ Sup. n. 72.

97 PROP. 3. A Parallelopipedon may also be conceived as composed of innumerable Planes all equal, and parallel to that which is its Base.

98 COROLLARY. It is the same of Cylinders, and Prisms.

99 PROP. 4. A Pyramid may be conceived as made by its Base, moving always Parallel to itself, and decreasing in proportion as it rises. If the Motion be according to its perpendicular Height, the Pyramid is a right Angled one; if it move Obliquely, it is an Oblique one.

100 PROP. 5. It may be conceived that a Pyramid is composed of innumerable parallel Planes, which diminish in Proportion to their Height.

101 PROP. 6. Two Solids having the same Height contain an equal number of Planes, that is, there may be conceived as many in one as in the other.

102 PROP. 7. If an Oblique Parallelopipedon, or Oblique Prism, every Pyramid, every Cone, are conceived as composed of innumerable Planes, their Surfaces are even.

CONSIDER the Pyramid BAC. If the Planes O, O, O, O, O, O, which compose it are thick, it is evident that its Surface is not even, but by Degrees so much greater, as the Planes O are thicker; and they are so much lesser, as there are more of them. If it is then supposed that they are infinite in Number, they would have no thickness; and then the Surface of BAC is without Degrees. Which is to be conceived of every Parallelopipedon, Prism, Pyramid and Cone.

103 PROP. 8. A Parallelopipedon may be conceived as described upon a given Line, Similar, and in the same manner posited as another given Parallelopipedon. Eucl. 11. Prop. 27.

EUCLID proposes the doing this Thing, (which is easy,) but it is sufficient to conceive it as being done.

104 PROP. 9. Two Solids are equal, which are composed of an equal Number of Planes equally thick, similar and equal.

IF for Example the two Solids X and Z were each composed of a Million of Planes equal and of an equal thickness, they are evidently equal. I add the Word Similar, to shew that in case the Planes which compose X ascend and diminish as those of Z arising in like manner diminish, each being Similar to its corresponding one, that is, the hundredth Part of X being equal and similar to the hundredth Part of Z, it must necessarily be that X and Z are equal.

105 PROP. 10. If the Parallelopipedon X be divided according to its Diagonal AC or EG, it is divided into two equal Triangular Prisms. Eucl. 11. Prop. 28. See P. B. Plate 9. Fig. 36. and 37.

THE two Parts of X have equal Bases, and the same Height: Therefore by the preceeding Proposition, they are equal. According

according to the Definition of Prisms, * they are Triangular ones.

PROP. 11. If the Sides of the opposite Planes of a Parallelopipedon are divided into two equal Parts, and that Planes be drawn thro' the Sections; the Line of common Section of the Planes, and the Diameter of the Parallelopipedon, are divided equally in two. Eucl. 11. Prop. 39.

THIS is evident, for the Diameter is the Base of a Triangle cut Parallel to one of its Sides thro' the Middle of the other; which divides the Side and the Base equally.

PROP. 12. If a Solid is bounded by parallel Planes, the opposite Planes are similar and equal Parallelograms. Eucl. 11. Prop. 24. (same figure)

1. THE Lines AD and BC are Parallel, † as also AB and DC. It is the same of the Lines EF and HG, and of FG and EH. Therefore the Planes ABCD and EFGH are Parallelograms. 2. These Parallelograms are Similar; for the Angle EFG is equal to ADC: ‡ &c. 3. The Parallels AB and HE between the Parallels AE and BH are equal, whether they be Oblique or Perpendicular. § Therefore it is demonstrable that the opposite Planes are equal.

THEOR. 1. EVERY Section of a Parallelopipedon, Prism, Cylinder, Pyramid, Cone, which is made Parallel to its Base, is Similar to its Base. Eucl. 11. Prop. 25.

1. IN the Parallelopipedon, Prism, and Cylinder, which are made by their Bases moving always Parallel, this is evident. 2. Also in the Pyramid, and Cone, which are Solids made by the Motion of their Base, which diminishes Proportionably, therefore the parallel Planes of which either of them may be conceived to be made, are all Similar. Now the Sections are some one of these Planes; consequently they are Similar to the Base; which was to be proved. For a farther Proof hereof, see P. B. Plate 9. Fig. 35. Which shews a real Parallelopipedon, cut Parallel to its Base.

THEOR. 2. ALL Solids of the same Name, (having the same Height, and equal Bases,) are equal, whether they be right, or Oblique. Eucl. 11. Prop. 29. 30. & 31.

THAT is, that Solid Parallelopipedons being upon equal Bases, and of the same Height, are equal one to another, they have an equal Number of Planes all similar and equal to their Bases; || therefore they ought to be equal. It is the same of Prisms, Cylinders, Pyramids, and Cones; for each Plane of one is equal and similar to each of the other taken at the same Height.

COROLLARY. THEREFORE in measuring Solids regard is had only to their Height, and Base.

Y

For

* Sup. n. 34. † Sup. n. 34. ‡ Sup. n. 36.
§ B. 2. n. 110. || Sup. n. 101.

For an Oblique Parallelopipedon hath a greater Surface than a right one. Notwithstanding if it be upon an equal Base, and of the same Height, it is not greater in its solid Content.

- 111 THEOR. 3. SOLID Parallelopipedons and Prisms having the same Height, are one to another as their Bases; and if they have the same Base, they are as their Heights. Eucl. 11. Prop. 32.

LET the two Solids be Z and Y. 1. They have an equal number of Planes, Parallel and Similar to their Bases. If their Bases are equal, then they are equal; if that of Z is double that of X, then all the Planes of Z being double those of X, Z will be the double of X. 2. If the two Solids have equal Bases, they are as the Heights, for all their Planes being equal, if the Solid Z is twice as high as X, it ought to have twice as many Planes. If it is three times higher, it will have three times as many.

- 112 THEOR. 4. Two Cylinders of the same Height are as their Bases, or if they have one and the same Base, they are as their Heights. Eucl. 12. Prop. 11 and 14.

THIS has been demonstrated of the Prism, and consequently of the Cylinder, which may be taken for a Prism.

- 113 THEOR. 5. If a Cylinder is divided by a Plane parallel to the Planes opposite to its Bases, the Segments of the Cylinder are one to another as the Segments of its Axis. Eucl. 12. Prop. 13.

THE Planes of the Sections are equal. * therefore all the Segments are as so many Cylinders whose Bases are equal, and therefore, by the preceeding Theorem, they are one to another as their Heights, which are the Segments of the Axis.

- 114 THEOR. 6. A Parallelopipedon is double a Triangular Prism of the same Height, if the Base of the Parallelogram is double that of the triangular Prism. Eucl. 11. Prop. 40.

THIS is evident, seeing that the two Prisms are as their Bases. † See P. B. Plate 9. Fig. 38 and 39.

- 115 THEOR. 7. EVERY Polygonal Prism may be divided into triangular Prisms.

LET K be a Polygonal Prism whose Bases are ABCDE and GHILF: Reduce the Polygonal Bases into Triangles. By the Definition of triangular Prisms, the Solids ABCGHI, ACDGIL, ADEGLE are triangular Prisms: therefore the Prism K may be divided into triangular Prisms.

- 116 THEOR. 8. A Prism is equal to several Prisms having the same Height, if its Base is equal to those of all the Prisms. It is the same of Pyramids.

FOR let there be conceived in these Solids Planes parallel to the Base, ‡ there will be an equal number of Planes in each; § the Planes in the greatest Prism are each equal to all the Planes which are in the other Prisms; for it is to them as its Base is to all

* Sup. n. 98. † Sup. n. 111. ‡ Sup. n. 97.
§ Sup. n. 101.

all the Bases of those Prisms. Now it is equal thereto, therefore, &c. It is the same of Pyramids.

COROLLARY. 1. A Cylinder is equal to a triangular Prism of the same Height, whose Base is equal to that of the Cylinder. 117

A Cylinder is a Polygonal Prism. Every polygonal Prism may be divided into triangular Prisms. * these Prisms are equal to one only triangular Prism of the same Height, whose Base is equal to all those of the Prisms. † Therefore the Cylinder equal to the polygonal Prism, is so likewise to the triangular Prism which hath the same Height, and whose Base is equal thereto.

COROL. 2. THEREFORE a Cylinder X is equal to several Cylinders A, B, C, &c. of the same Height, whose Bases all taken together are equal thereto. 118

For all these Cylinders are equal to so many triangular Prisms, having the same Height and equal Bases. That to which X is equal, (and therefore of the same Height and upon an equal Base,) is equal to all the other triangular Prisms, and therefore to the Cylinders A, B, C, &c. whose Bases all together are equal to that of X; wherefore X is equal to A, B, C, &c.

THEOR. 9. A triangular Prism may be divided into three equal triangular Pyramids. Eucl. 12. Prop. 7. 119

LET X be a triangular Prism; upon each of its three Sides draw Diagonals, which make six Triangles. The Triangles BAD, BCD are equal, † the Pyramids BADF and BDCF which have the same Vertex, and therefore the same Height are equal. § these two Pyramids, being conceived as deducted from the Prism X, there remains a third viz. FCED, which hath in the first Place the same Vertex D, and therefore the same Height as the Pyramid FBCD; they have equal Bases, viz. the equal Triangles EBC and FCE: || Therefore they are equal. * Now the Pyramid FBCD is the same as BDCF, being formed by the same Triangles: therefore the Pyramids FCED and BADF are equal, being so to a Third; therefore the Prism X may be divided into three equal Pyramids, which are BADF, BDCF, and CEDF. See also P. B. Plate 10. Fig. 41.

COROL. 1. THEREFORE every Pyramid is a third Part of the whole Prism of the same Height, which is upon the same, or an equal Base. 120

COROL. 2. THEREFORE to measure a Pyramid, the Base must be multiplied by one Third of its height. 121

LEMMA. A Polygonal Pyramid may be divided into triangular Pyramids. 122

THE Base of a polygonal Pyramid is a Polygon, consequently it may be reduced into Triangles, upon which Planes being conceived to as raised along the Sides of that Pyramid even to

* Sup. n. 115. † Sup. n. 116. ‡ B. 2. n. 128.
§ Sup. n. 109. || B. 2. n. 128.

* Sup. n. 109.

its Vertex, there will be several triangular Pyramids, which are the Parts of the Polygonal Pyramid.

- 123 THEOR. 10. PYRAMIDS having triangular Bases, and which are of the same Height, are one to another as their Bases. It is the same of those which have polygonal Bases. Eucl. 12. Prop. 5 and 6.

PRISMS are triple the Pyramids of the same Height and Base. But Prisms of the same Height are one to another as their Bases, or those of the same Base are as their Heights: † Therefore Pyramids which are only the one Third thereof, are so likewise, by the preceeding Lemma, seeing that polygonal Pyramids may be reduced into Triangular ones, they may be taken for each other promiscuously.

- 124 THEOR. 11. EVERY Pyramid having a Triangular Base may be divided into two equal Pyramids, similar one to another, and to the whole Pyramid, and into two equal Prisms, which two Prisms are together greater than the half of the whole Pyramid. Eucl. 12. Prop. 3.

- 125 THEOR. 12. Two Pyramids of the same Height having triangular Bases, may be divided into two other Pyramids equal one to another and Similar to the whole Pyramid, and into two equal Prisms; and if the Pyramids proceeding from this Division, are continually divided in the same Manner: As the Base of one of the Pyramids is to the Base of the other; so all the Prisms which are in one of the Pyramids, are to all the Prisms of the other being equal in Number. Eucl. 12. Prop. 4.

THE Demonstrations of these two Propositions are here omitted, as being difficult to conceive by the bare Representation thereof upon a Plane; I therefore refer the Reader to P. B. Plate 10. Fig. 40. Which contains a Pyramid exactly divided, according to the preceeding Theorem, from whence the other may be easily comprehended.

- 126 THEOR. 13. EVERY polygonal Pyramid is the third Part of the whole Prism of the same Height, and which is upon the same or an equal Base.

FOR having reduced the Base of each of these two Solids into Triangles, the polygonal Pyramid will be divided into triangular Pyramids. Now each of these triangular Pyramids, is the third Part of each of the triangular Prisms; ‡ the whole polygonal Pyramid is therefore the third Part of the whole polygonal Prism.

- 127 PROBLEM. To find the Solidity of the Frustum of a Pyramid, whose Bases are Parallel; let that of the Bottom be for Example 36, the top One 9, the Number 18 is a mean Proportional between these two Numbers.

RULE. These two Numbers added into one Sum, and that Sum multiplied by 2, which I suppose to be the one Third of the Frustum's Height, the Product 126 is the Frustum's Solidity.

To

* Sup. n. 120.

† Sup. n. 111.

‡ Sup. n. 120.

To prove the Truth hereof; let the bottom Base be a , the top One b , and c the Height of the Frustum Z , let the Part of the Axis which is wanting to compleat the Pyramid be d , the Solidity of a Prism which is triple the whole Pyramid, is $aac + aad$, from which I take bbd the Solidity of a Prism, whose third Part is the Pyramid X : Hence $aac + aad - bbd = 3Z$. Seeing that $\frac{aa}{ab} = \frac{bb}{bb}$; then, according to the Rule, these three Terms which are the same as 36, 18, 9, being multiplied by the Third of c , will give the Solidity of Z ; so multiplying them by the whole c , I shall have $aac + abc + bbc = 3Z$: it remains to be demonstrated that $aac + aad - bbd = aac + abc + bbc$: Take away aac from each part, there remains $aad - bbd = abc + bbc$. it must be shewn that this is so. $a - b$. $b :: c$. d multiplying $a - b$, and b by $a + b$, * $aa - bb$. $ab + bb :: a - b$. b . therefore $aa - bb$. $ab + bb :: c$. d : $\dagger$ then multiplying the Extremes and the Means, there is $aad - bbd = abc + bbc$; $\dagger$ which remained to be proved. The Rule is therefore true. The true Form of the Frustum of a Pyramid, is exhibited in P. B. Plate 7. Fig. 26.

THEOR. 14. A Cone is the Third of a Cylinder of the same Height their Bases being equal. Eucl. 12. Prop. 10. 128

A Cone is a Pyramid of infinite Sides: Now a Pyramid is the third Part of a Prism of the same Height, having an equal Base: & therefore the Cone is also the third Part of a Cylinder of the same Height, and upon the same or an equal Base, seeing that a Cylinder is a Prism having an infinite number of Sides.

COROLLARY 1. A Cone is equal to all those Cones which have the same Height, and whose Bases all taken together are equal to the Base thereof. 129

CONES are Pyramids of an infinite number of Sides, which constitute the third Part of the Cylinder or Prism; therefore this Proposition is the same as that which has been proposed, ||

COROL. 2. Two Cones of the same Height are as their Bases; and if they have the same Base, they are as their Heights, which is evident. Euclid 12. Prop. 11 and 14. 130

THEOR. 15. SIMILAR Prisms and Parallelopipedons are in a Ratio compounded of the Ratio's of their three Dimensions, and that Ratio is triple. Eucl. 11. Prop. 33. 131

THEIR Solidity depends upon the Multiplication of their Dimensions. * The Ratio which they have one to another is compounded of that of their three Dimensions. $\dagger$ Which being similar, the Ratio is triple. $\dagger$

THEOR. 16. SIMILAR Cylinders are one to another in Ratio compounded of their Dimensions, and that Ratio is triple. Eucl. 12. Prop. 12. 132

For

* B. 3. n. 54. $\dagger$ B. 3. n. 53. $\dagger$ B. 3. n. 56.

$\S$ Sup. n. 120. || Sup. n. 116.

* Sup. n. 110. $\dagger$ B. 3. n. 78. $\dagger$ B. 3. n. 74.

FOR Cylinders being as Prisms having Bases of an infinite Number of Sides, they are one to another as these Prisms, by the preceeding Theorem.

- ¹³³ THEOREM 17. SIMILAR Pyramids are in Ratio compounded of the Ratio's of their three Dimensions, and that Ratio is triple. Eucl. 12. Prop. 8.

THIS is evident, for they are the third Part of the Prisms which are upon their Bases, and which have the same Height, * and therefore in the same Ratio.

- ¹³⁴ THEOR. 18. SIMILAR Cones are one to another in Ratio compounded of the Ratio's of their Dimensions, and that Ratio is triple. Eucl. 12. Prop. 12.

THIS is a Consequence which naturally follows, from the Similarity which is between the Cone and a Pyramid, whose Base has an infinite Number of Sides.

- ¹³⁵ THEOR. 19. SIMILAR Cylinders are one to another as the Cubes of the Diameters of their Bases.

THEY are in Ratio triple that of each of their Dimensions, and consequently of the Ratio of the Diameter of their Bases. Now the Cubes of their Diameters are in Ratio triple that of the same Diameters. † therefore seeing that Ratio's compounded of equal Ratio's are equal, similar Cylinders are one to another as the Cubes of the Diameters of their Bases. It is the same of similar Cones, which is demonstrable in the same manner.

- ¹³⁶ THEOR. 20. EQUAL Parallelopipedons have their Heights and Bases reciprocal; and if they are reciprocal, the two Solids are equal. Eucl. 11. Prop. 34.

LET X and Z be two equal Parallelopipedons, A the Height of X, and B that of Z, let the Content of the Base of X be M, and N that of the Base of Z. According to what is supposed, $A \times M = B \times N$; then $A, B :: N, M$; ‡ wherefore these four Quantities A, B, N, M are reciprocal. § Now if these four Quantities are reciprocal, that is, If A, M, B, N may be so ranged that $A, B :: N, M$, then will $A \times M = B \times N$. ||

- ¹³⁷ THEOR. 21. Two Cylinders being equal, (It is the same of Pyramids and Cones,) their Heights and Bases are reciprocal; and if they are reciprocal, the two Cylinders are equal. Eucl. 12. Prop. 9 and 15.

THE Demonstration of the preceeding Theorem will serve for this; by only supposing X and Z to be two Cylinders.

- ¹³⁸ THEOR. 22. IF three right Lines A, B, C, are proportional, the Solid Parallelopipedon ABC composed of these three Lines, is equal to the Parallelopipedon BBB made of the mean Term B, provided that these Solids are Equiangular. Eucl. 11. Prop. 36.

∴ A. B.

* Sup. n. 120. † B. 3. n. 87. ‡ B. 3. n. 58.

§ B. n. 3. 61.

|| B. 3. n. 56.

∴ A. B. C. then $AC=BB$.† then AC and BB multiplied by B, the Product ACB is also equal to the Product BBB.§ Now ACB is the Parallelopipedon made of the three Lines A, B, C, equal therefore to BBB which is made of the mean Line B.

THEO. 23. A, B, C, D, are four proportional Lines. The similar Solids constructed upon these Lines are proportional; and if they are in Proportion, the four Lines are proportional. Eucl. 11. Prop. 37.

THIS has been proved.¶

THEO. 24. A Sphere is equal to a Cone, or Polygonal Pyramid, whose Axis is the Radius of that Sphere, and its Base a Circle, whose Radius is the Diameter of that same Sphere.

1. BY conceiving infinite Cones or polygonal Pyramids, whose Vertices are in the Center of the Sphere, and the Bases in the Surface of the same Sphere; it is evident that it may be said, that the Solidity of that Sphere is equal to all those Cones or polygonal Pyramids, seeing 'tis allowed, that the whole is equal to all its Parts taken together. 2. All these Cones are equal to a Cone having the same Height, (viz. the radius of that Sphere,) and for its Base the Surface of that Sphere, (which is equal to the Bases of these Cones.)* Now the Surface of that Sphere is equal to that of a Circle, whose radius is the diameter of that Sphere: † the Solidity is therefore equal to that of a Cone whose Base, &c. Suppose the ratio of the diameter of a Circle to its Circumference be as 7 to 22, it may be said that the Solidity of the Sphere is to the Cube of its diameter as 11 to 21: for let m be the Circumference of the Circle, and n the diameter. 1. mn . mn . m . n . ‡ Now m is to n as 22 to 7, or 66 to 21. therefore the sixth Part of mn , which is the Solidity of the Sphere, (as is easily perceived from what has been heretofore demonstrated,) is to mn the Cube of its Diameter, as the sixth Part of 66, that is, as 11 is to 21; which was to be proved.

THEOREM 25. THE ratio of the Cylinder X to its inscribed Sphere Z, is sesquialter.

LET B and C be two Cones whose Axis is the radius of the Sphere Z, and that the radius of the Base of B be that of the Sphere Z, and the radius of the Base of C be the Axis or diameter of the Sphere X, then these two Cones B and C are one to another as their Bases. § Now that of C is quadruple that of B. therefore the Cone C is quadruple the Cone B; hence B. C.: 1. 4. the lesser Cone B is the sixth Part of the Cylinder X, whose Base is the greatest Circle of the Sphere Z, and its Axis the diameter thereof; for the Cone B is the third Part of a Cylinder, having the same Base and Axis; || consequently it is the sixth Part of a Cylinder having twice that Axis, and the same Base; hence X. B.: 6. 1. The Cone C is equal to the Sphere

† B. 3. n. 37.

§ B. 3. n. 54.

¶ B. 3. n. 85.

* Sup. n. 129.

† Sup. n. 91.

‡ B. 3. n. 54.

§ Sup. n. 130.

|| Sup. n. 128.

Sphere Z, ¶ it has been proved that B. C:: 1. 4; hence B. Z:: 1. 4: therefore seeing that the Cylinder X contains six such Parts, as the Sphere X contains four, X. Z:: 6. 4; which is a sesquialter Ratio.

- 143 THEOREM 26. SPHERES are one to another as the Cubes of their Diameters, or in Ratio triple that of their Diameters. Eucl. 12. Prop. 18.

SPHERES are in a Ratio compounded of the Ratio's of their three Dimensions; all the Parts of Spheres are similar; therefore there three Dimensions have the same Ratio; therefore the Ratio which they compose is triple that of each of the Ratio's of their Dimensions; for example, of that of their Diameters. Now the Cubes of the Diameters are in triple Ratio of that of these Diameters: Spheres are then one to another as the Cubes of their Diameters.

S E C T. V.

Of the Method of inscribing or circumscribing a Sphere with the five regular Bodies.

ADVERTISEMENT.

- 144 THE true Forms of those Bodies are shewn in P. B. Plate I. Fig. 1, 3, 4, 5, 6.

- 145 THEOREM 1. EVERY Section of a Sphere by a Plane, is a Circle.

X is the Section of a Sphere whose Center is A; it must be proved that this Section is a Circle: to do, which let us conceive, 1. That a Perpendicular AB is let fall from A the Center of the Sphere upon the Plane of this Section, which I call X. 2. That Lines, such as AC, be drawn from the same Center A to all the extreme Parts of X: these Lines (which are all Radii of the Sphere,) are equal, they are Oblique, by reason there can be drawn but one Perpendicular from A to X. Now oblique Lines which are equal have their Bottoms equally distant from the Perpendicular,* therefore all the Lines drawn from the Extremities of X to the Point B are equal, and the Extremities are consequently in the Circumference of a Circle; therefore X is a Circle, according to the Definition thereof.†

- 146 PROBLEM 1. To inscribe a regular Polygon having even an Number of Sides, in the greatest of two unequal concentric Circles, so that the Polygon may not touch the lesser Circle. Eucl. 12. Prop. 16.

PROP.

¶ Sup. n. 140.

* B. 1. n. 61,

† B. 1. n. 20,

PROB. 2. To inscribe in the greatest of two unequal Spheres¹⁴⁷ having one and the same Center, a Polyhedron whose Planes touch not the Surface of the lesser Sphere. Eucl. 12. Prop. 17.

THESE TWO Problems appear to me as unnecessary, therefore I pass them.

LEMMA. 1. IF aa , the Square of AC , is triple of bb , the¹⁴⁸ Square of CD a mean proportional between AD and BD , I say that DB is the third Part of the diameter AB .

LET AD be called c , 1. $aa=bb+cc$,* and because that by the Supposition $3bb=aa$: then $3bb=bb+cc$: taking away bb from each side, it will be $2bb=cc$. 2. $\therefore AD \cdot CD \cdot BD$. or $\therefore c \cdot b \cdot BD$ by the Hypothesis: then cc or $2bb$. $bb \div \therefore AD \cdot DB$.† therefore AD is double BD , and consequently DB is the third of AB ; which was to be proved.

THEOR. 2. THE Square of one of the Sides of a Tetraedron, or¹⁴⁹ Equilateral Pyramid, is equal to six Times the Square of the third Part of the diameter of its circumscribing Sphere.

THE Tetraedron or Equilateral Pyramid, is composed of four equal and Equilateral Triangles. Let us imagine a Tetraedron to be inscribed in the Sphere X , one of whose Sides is AC , or a , and that CD is the Radius of the Circle in which one of the

Equilateral Triangles which compose the Solid is inscribed: AC

$= 3 CD$, or $aa=3bb$;‡ consequently DB is the third Part of AB the Diameter of the Sphere X , by the preceeding Lemma. Then let $BD=c$, and consequently $AB=3c$. Since that $\therefore 3c$. $a \cdot 2c$:§ then $6cc=aa$;|| Q E. D.

COROLL. THE Square of the diameter of a Sphere, is in sesqui-¹⁵⁰alter ratio to the Square of one of the Sides of its inscribed Tetraedron. Eucl. 13. Prop. 13.

IT has been proved that aa , the Square of the Tetraedron's Side, is equal to six Times the Square of c , the third Part of the diameter of the Sphere. Now the Square of $3c$, the diameter of the Sphere is $9cc$; therefore the ratio of the side of the Tetraedron to that of the diameter of the Sphere is as $6cc$ to $9cc$, or 6 to 9, which is a sesquialteral Ratio.

THEOR. 3. THE side of the Tetraedron is incommensurable in it-¹⁵¹self, and commensurable in Power, with the diameter of its circumscribing Sphere.

LET AC or a , be the side of the Tetraedron as above, and AB or $3c$ the diameter of the Sphere, by what has been demonstrated in the preceeding Theorem, DB is equal to c , and $AD=2c$; hence $\therefore 3c$. $a \cdot 2c$:¶ therefore $9cc$. aa :‡ $3c$. $2c$.* Now 3 and 2 are not square Numbers; therefore a is incommensurable in itself with $3c$, and commensurable in Power.† In the preceed-

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ing

* B. 4. n. 78.

† B. 3. n. 86.

‡ B. 4. n. 146.

§ B. 4. n. 28.

|| B. 3. n. 57.

¶ B. 4. n. 28.

‡ B. 3. n. 86.

† B. 4. n. 115.

ing Theorem, we proved that aa is to the Square of the Sphere's Diameter as 6 to 9.

- 152 PROB. 3. To inscribe a Tetraedron in a Sphere, or to find a Circle sufficient to contain one of the Sides of the Tetraedron. Eucl. 13. Prop. 13.

DIVIDE AB the diameter of the Sphere into three equal Parts, so that DB be double AD , and upon D erect the Perpendicular DC , which is the Radius of a Circle, in which an Equilateral Triangle being made whose side is BC , you will have one of the Sides of a Tetraedron, as is evident.*

- 153 THEOR. 4. THE Square of the diameter of a Sphere is triple the Square of each side of its inscribed Cube, or Hexaedron. Eucl. 13. Prop. 15.

THE Cube or Hexaedron X is inscribed in a Sphere. Let the diagonal $AB = m$, which is the diameter of the Sphere, let the diagonal of one of the Sides of the Cube be BD . I call s all

the Sides of the Cube, which are all equal. $AB = BD + AD$,

or $mm = nn + ss$, † so likewise $BD = BC + CD$, or $nn + ss$; then substituting in the room of nn its equal ss , it will be $mm = 3ss$.

Q. E. D.

- 154 PROB. 4. THE diameter of a Sphere being given, to find the side of the Cube or Hexaedron which may be inscribed therein, or to find a Circle which may contain one of the Sides of the Cube. Eucl. 13. Prop. 15.

LET AB be the diameter of a Sphere in which must be inscribed a Cube, divide it into three such Parts, that BD be double of AD : upon D raise the Perpendicular CD , and from C draw a Line to A , which is the side of the Cube required. For let $BA = 3c$, and $CA = d$: then $\div 3c. d. c.$ § then $3cc = dd$. || the Square of AB or $3c$ is $9cc$; therefore the Square of AB is triple that of d , which is only $3cc$. Wherefore AC is the side of the Cube required, by the preceeding Theorem. Then if you would have a Circle capable of containing one of the Sides of the Cube, there must be a Square described one of whose Sides is AC , and its circumscribing Circle will be that which was required.

- 155 THEOR. 5. THE side of a Cube is incommensurable in itself, and commensurable in Power, with the diameter of a Sphere. (same Figure.)

AC or d , the side of the Cube, is a mean proportional between the whole diameter AB or $3c$, and its third Part c ; hence since that $\div 3c. d. c$; then $3c. c :: 3. 1$. the two Numbers 3 and 1 are not square Numbers, therefore AC is in itself incommensurable with AB ; but commensurable in Power, by reason its square is the third Part of AB . ¶

THEOR.

* Sup. n. 149.
|| B. 3. n. 57.

† B. 4. n. 78.
¶ B. 4. n. 131.

§ B. 4. n. 28.

THEOR. 6. THE Square of each side of an Octoedron is half that of the diameter of its circumscribing Sphere. Eucl. 13. Prop. 14.

AN Octoedron is composed of eight equal Equilateral Triangles, whose Sides are Chords of a Quadrant of a Circle or ninety Degrees. Now it is evident, that the Square of the Chord of ninety Degrees is half that of the Diameter: for two Chords of ninety Degrees make a right Angle, whose Base is the diameter of the Circle; therefore the Square of these two Chords is equal to that of the Diameter, which is consequently double that of each of the two Chords.

THEOR. 7. THE same Circle will contain the Square which is one of the Sides of a Cube, and the Triangle which is one of the Sides of an Octoedron, they being both inscribed in the same Sphere. Eucl. 14. Prop. 8.

LET the Square BCDE be the side of a Cube, inscribed in a Circle, the side of an Octoedron inscribed in a Circle is the Triangle FGH. It must be demonstrated that if these two Solids are inscribed in one and the same Sphere, these two Circles are equal. Let a be the diameter of the Sphere, b the side of the Square which is one of the Sides of the Cube, and c the side of the Triangle which is one of the Sides of the Octoedron. Let the Radius of that Circle which comprehends the side of the Cube be called y , and that of the Circle which contains the side of the Octoedron be x . It must be proved that $y=x$. 1. If bb is conceived to be the Square of the Cube inscribed in the Circle whose Radius is y , it is evident that $2yy=bb$.* and since that $3bb=aa$:† then $aa=6yy$. 2. $3xx=cc$.‡ Now $2cc=aa$:|| then $6xx=aa$. hence since $6yy=aa=6xx$, then $6yy=6xx$; therefore $y=x$; which was to be proved.

THEOR. 8. THE side of an Octoedron is incommensurable in itself, and commensurable in Power with the diameter of its circumscribing Sphere.

THE Square of each side of the Octoedron is to that of the diameter of the Sphere, as 1 to 2.¶ Now 1 and 2 are not square Numbers; therefore the side is incommensurable in itself, with the diameter of the Sphere,* and commensurable in Power, by reason its Square is half that of the Sphere.

PROP. 5. TO find the side of an Octoedron, and a Circle capable of containing one of the Sides thereof.

FIND by the fourth Problem a Circle that will comprehend one of the Sides of a Cube, that Circle by the seventh Theorem, is sufficient to contain one of the Sides of the Octoedron, there is then nothing required to be done, but to inscribe an Equilateral Triangle in that Circle: the side of that Triangle is the side of the Octoedron required.

THEOR. 9. DIAGONALS or Chords being drawn upon the twelve Pentagons which compose the Dodecaedron, those Diagonals which

Z 2

join

* B. 4. n. 78.
§ Sup. n. 156.

† Sup. n. 153.
|| Sup. n. 156.

‡ B. 4. n. 146.
* B. 4. n. 115.

join together form six Squares which are the six Sides of a Cube or Hexaedron inscribed in the same Sphere as the Dodecaedron.

LET there be four pentagonal Sides of a Dodecaedron, K, X, Z, Y, (in reading this keep in hand a Dodecaedron) upon each of these Sides conceive Diagonals, one from A to B, from B to C, from C to D, and from D to A, so upon all the other Sides; it must be proved. 1. That these four Diagonals make a Square, viz. ABCD. 2. That the other Diagonals with these, form six Squares equal to ABCD, which compose a Cube inscribed in the same Sphere as the Dodecaedron; therefore each side of the Cube is equal to each diagonal of the Pentagons. 1. All these Diagonals subtending equal Angles, are equal. 2. Let us conceive Lines to be drawn from the four Points A. B. C. D. which are upon the Sphere in which the Dodecaedron is inscribed to the Center of the Sphere, they will make a quadrilateral Pyramid, whose Base is the plane which divides the Sphere and passeth thro' these four Points. 3. The Section of the Sphere by the Plane ABCD is a Circle.* Now there can be no Figure of four equal Sides inscribed in a Circle, except the Square; for the four Angles are equal to four right ones: † and by reason that they stand upon equal Arcs, they are equal: therefore the Figure ABCD, which hath equal Sides, and which is inscribed in a Circle, is a Square. 4. Every Pentagon may be reduced into three Triangles; wherefore the Surface of a Dodecaedron, consisting of twelve Pentagons, is reducible into thirty-six Triangles. Now every Square equal to ABCD subtends six, as appears in the Figure; therefore the thirty-six Triangles are subtended by only six equal Squares, which compose a Cube inscribed in the same Sphere; and therefore it may be truly said that the diagonal of a Pentagon, which is one of the Sides of a Dodecaedron inscribed in a Sphere, is equal to a side of a Cube inscribed in the same Sphere.

161 PROB. 6. To find the side of a Dodecaedron, and a Circle sufficient to contain one of the Sides thereof. Eucl. 13. Prop. 17.

In the first Place find the side of a Cube inscribed in the proposed Sphere: ‡ this side is equal to the diagonal of each pentagonal Face of the Dodecaedron: § divide this side into mean and extreme Ratio: its greater Part is the side of the Dodecaedron proposed || To find the Circle sufficient to contain one of the Sides of the Dodecaedron, describe a Pentagon as has been taught, ¶ then circumscribe it with a Circle, and it will be that which was required.

162 THEOR. 10. The side of a Dodecaedron is incommensurable with the diameter of the Sphere, both in itself and in the second Power. Eucl. 13. Prop. 17.

1. LET the diameter of the Sphere be b , that of the side of the Cube inscribed in the Sphere be $c+d$, divided into mean and extreme Ratio, whose greater Part c is the side of the Dodecaedron.

* Sup. n. 145.
§ Sup. n. 160.

† B. 2. n. 113.
|| B. 4. n. 153.

‡ Sup. n. 154.
¶ B. 4. n. 159.

caedron. * 2. c is incommensurable, with $c+d$ both in itself and in Power. † 3. $c+d$ is commensurable with b in the second Power, ‡ that is, $cc+2cd+dd$ is commensurable with bb . cc is then incommensurable with bb ; for if it were commensurable with bb , it would be so with the Square of $c+d$, § which bb is commensurable to, seeing it is the Triple of the Square, || consequently cc incommensurable in power with bb , is also incommensurable in itself with b . †

LEMMA 2. *MN is the Diameter of a Circle, in which the two Chords AB and CE which intersect MN at right Angles, are Parallel one to another, and the Distance of FG equal to the half of each: I say that MF is the Side of a Decagon inscribed in a Circle, whose Radius is FA.* 163

SUPPOSING MF or $GN = x$, and $AF = z+x$: if MF or x is the Side of a Decagon whose Radius is AF or $z+x$; then having divided AF into mean and extreme Ratio, x is the mean Part, ¶ and consequently $z+x : x :: x : z$; therefore if we demonstrate that $z+x : x :: x : z$ we shall do what is proposed. Seeing $AF = z+x$; then $AB = 2z+2x$, and $BC = z+x$. The Square of AB , which is $4xz+8zx+4xx$, with that of BC , which is $zz+2zx+xx$ are equal to that of AC or MN . † Now by the Hypothesis $MN = 3x+z$; for MF and GN are each equal to x , and $FG = z+x$: The Square, I say of $3x+z$ is $9xx+6xz+zz$: then putting the Squares of AB and BC both into one Sum, $9xx+6xz+zz = 5zz+10xz+5xx$; taking away $5xx+6xz+zz$, from each part, there will remain $4xx = 4zz+4xz$, dividing both by 4, it will be $xx = zz+zx$; then resolving this Equation into a Proportion ‡ it will be $z+x : x :: x : z$, seeing that the Product of the Extremes $z+x$ and z , which is $zz+zx$, is equal to xx the Square of the mean Term x ; Q. E. D.

LEMMA 3. *The Line AM is the Side of a Pentagon inscribed in a Circle, whose Radius is AF. (same fig.)* 164

MF is the Side of a Decagon inscribed in a Circle, whose Radius is AF , § the Square of AF together with that of MF is equal to that of AM ; therefore AM is the Side of a Pentagon. ||

LEMMA 4. *The Square of AF, or BF, or GE, or GC, which are equal Lines, is the fifth Part of that of the Diameter AC, or MN. (same fig.)* 165

LET $AF = b$; then $AB = 2b$, and $BC = b$, the Square of AB is $4bb$, and that of BC is bb . Now both the Squares which make $5bb$, are equal to that of AC or of MN : * therefore it is equal to five times that of AF .

Lemma

* B. 4. n. 153. † B. 4. n. 140. ‡ Sup. n. 155.

§ B. 4. n. 113. || Sup. n. 153.

† B. 4. n. 115. ¶ B. 4. n. 151. † B. 4. n. 78.

‡ B. 3. n. 58. § Sup. n. 163.

|| B. 4. n. 161. * B. 4. n. 78.

166 LEMMA 5. To find a Line whose Square is the fifth Part of that of MN, the Diameter of the given Circle X.

AM is the fifth Part of the Square of MN, the Square of MN is five times that of MB; † hence MB is the Line that was required, that is, equal to AF of the preceeding Figure, seeing the Square of AF is the fifth Part of MN. †

167 PROBLEM 7. MN the Diameter of a Sphere being given, to describe an Icosaedron. Eucl. 13. Prop. 16.

1. HAVING found the Length of AF, (see the Figure of Lemma the 2d. Number 163,) and having divided it equally in two: From the Center make DF and DG equal to the Half thereof, so that $FG=AF$; then draw AB and CE, intersecting MN at right Angles. 2. Taking AB and CE for Diameters, describe two Circles which I call Z and Y, which are Parallel, (see the following Figure,) being upon Planes suppoed to be Parallel. 3. In each of these two Circles inscribe a Pentagon, and from each of its Angles draw right Lines to M and to N the Ends of the Diameter of the Sphere; which make five Triangles, whose Sides are each equal to the Side of the Pentagon inscribed in these two Circles, by Lemma the Third; hence all the Sides of these Triangles being equal to the Sides of Pentagons, they form two solid Angles upon the Circles Z and X, each consisting of five Equilateral Triangles, whose Vertices are at the Extremes of M and N of the Diameter of the Sphere, and therefore here is found ten Sides of the Icosaedron. Neither the Sides, nor solid Angle whose Vertex is N, are marked with Letters for fear of rendering the Figure confused, they must therefore be conceived by the Imagination. 4. Also in the same Circles Z and X, inscribe a Decagon, whose corresponding Angles in X and Z join by the Lines BE, PK, IH, DG, FL, &c. which by the Hypothesis are all equal to the Radii of Z and X. 5. Draw the Diagonals BK, KI, IG, GF, &c. The Square BP, the Side of the Decagon, and that of PK, which is equal to the Radius of Z and X, are together equal to that of BK; § therefore BK is the Side of a Pentagon inscribed in X and Z. || The same Thing is demonstrable of KI, of IG, of GF, &c. The Triangles BKL, KIG, ICF, &c. have for their Bases the Sides of the said Pentagon; they are therefore Equilateral one to another, and to the Ten which compose the two solid Angles, (which we treated of above.) consequently there are between Z and X ten of these Triangles, five of which have their Bases upon Z, and the other Five upon X, which with the Ten already found make the Twenty equal and equilateral Triangles which ought to constitute the Icosaedron; which was to be done.

Corollary

† B. 4. n. 63.

‡ Sup. n. 165.

§ B. 4. n. 78.

|| B. 4. n. 164.

COROLLARY 1. THE Square of the Diameter of the Sphere is¹⁶⁸ quintuple the Square of the Radius of the Circle X or Z, which is the Base of a Solid Angle compos'd of five Equilateral ones: As has been demonstrated in the Theorem, and elsewhere. *

COROL. 2. THE Diameter MN is compounded of the Side of¹⁶⁹ the Hexagon, or Radii of the Circles Z and X, and of two Sides of the Decagon inscribed in these Circles. As has been demonstrated both in the Theorem, and before. †

COROL. 3. THE Sides of the Triangles of the Icosaedron are¹⁷⁰ equal to the Sides of the Pentagons inscribed in Z or X.

THEOR. 11. THE Sides of the Icosaedron are incommensura-¹⁷¹ ble, both in themselves and in Power, with the Diameter of its circumscribing Sphere. Eucl. 13. Prop. 16.

THE Square of the Radius of the Circles which are described in order to make the Icosaedron, is the fifth part of that of the Diameter of the Sphere. † Let the Square be bb , therefore that of the Diameter of the Sphere is $5bb$. These two Squares are then commensurable, they being as 1 to 5. Let x be the Side of the Triangles which constitute the Icosaedron, which x is one of the Sides of the Pentagon inscribed in a Circle whose Radius is b , § wherefore bb and xx the Squares of the Side of the Pentagon whose Radius is b , are incommensurable, as well as their roots x and b . || and since that bb the Square of the Radius is commensurable with the Square of the Diameter of the Sphere, then xx is incommensurable with the Square of the Diameter; for if it were commensurable thereto, it would be so with that of b ; *and if xx and the Square of the Diameter are incommensurable, x and the Diameter are so likewise, seeing the Ratios of Squares are duple that of their Roots, and that therefore if the Duples are Surds, then the compounding Parts are so likewise: For the Product of two numbers is a Number.

THEOR. 12. THE same Circle comprehends the Pentagon which¹⁷² is one of the Sides of the Dodecaedron, and the equilateral Triangle which is one of the Sides of the Icosaedron. Eucl. 14. Prop. 3.

X and Z are two Circles, in which suppose the Pentagon of X to be one of the Sides of a Dodecaedron, and the equilateral Triangle of Z one of the Faces of an Icosaedron, and these two Bodies inscribed in one and the same Sphere whose Diameter is a . It must be proved that m , the Radius of X, is equal to n the Radius of Z. Describe the Pentagon Y having each Side equal to the Side of the Equilateral Triangle of the Icosaedron. The Pentagon is therefore the Base of five of the Triangles of the Icosaedron. † Divide DE the Radius of Y, at the Point G into mean and extreme Ratio. The greater Part DG is the Side of the Decagon. ‡ Let BC be likewise divided into

mean

* Sup. n. 165. † Sup. n. 163. ‡ Sup. n. 168.

§ Sup. n. 170. || B. 4. n. 170.

* B. 4. n. 113. † Sup. n. 170. ‡ B. 4. n. 151.

mean extreme Ratio at the Point K, the greater Part BK is equal to the Side of the Pentagon: * Therefore BC. DF::BK.

DG; † hence BC. DF::BK. DG. † And $\sqrt[3]{BC} \cdot \sqrt[5]{DF}::\sqrt[3]{BK}$.

$\sqrt[3]{DG}$. § Now BC is equal to the Side of the Hexaedron, || three Squares of which are equal to aa, (the Square of the Diameter of its circumscribing Sphere. * And $\sqrt[5]{DF}$ is also equal: to aa†

Then $\sqrt[3]{BC}=\sqrt[5]{DF}$. Hence there having been an Equality in the above Proportion, between $\sqrt[3]{BC}$ and $\sqrt[5]{DF}$. There is also

an Equality between $\sqrt[3]{BK}$ and $\sqrt[5]{DG}$: Then seeing that $FE=$

$DF+DG$; † then $\sqrt[5]{FE}=\sqrt[5]{DF+DG}$: Then also $\sqrt[5]{FE}=\sqrt[3]{BC}$

$+ \sqrt[3]{BK}$: For it has been shewn that $\sqrt[3]{BC}=\sqrt[5]{DF}$, and $\sqrt[3]{BK}=$

$=\sqrt[5]{DG}$, but $\sqrt[3]{BC}+\sqrt[3]{BK}=\sqrt[5]{mm}$, § and $\sqrt[5]{FE}=\sqrt[5]{13nn}$; || For

FE is supposed equal to the Side of the equilateral Triangle whose Radius is n, and consequently since $\sqrt[5]{15mm}=\sqrt[3]{BC}+\sqrt[3]{BK}=$

$\sqrt[5]{FE}=\sqrt[5]{15nn}$. Therefore $\sqrt[5]{15mm}=\sqrt[5]{15nn}$, or $mm=nn$, and consequently $m=n$; which was to be proved.

PROB. 8. THE Diameter of a Sphere being given, to find the Sides of the five regular Bodies which are inscribed therein. Eucl. 13. Prop. 18.

PROCEED as been taught, concerning the Finding of the Proportion of each Side of these regular Bodies to the Diameter of their circumscribing Sphere.

THEOR. 13. THE Superficies of a Dodecaedron is equal to thirty times the Rectangle comprehended under one of the Sides of the Pentagon, which is one of its Faces, and the Apotome of that Pentagon; and that of the Icosaedron, is equal to thirty times the Rectangle comprehended under one of the Sides of the equilateral Triangle, which is one of its Faces, and the Apotome of that Triangle. Eucl. 14. Prop. 4.

HAVING drawn Lines from the Center of the Pentagon and from that of the equilateral Triangle to their angular Points, X will be thereby divided into five Triangles, and Z into three; hence the twelve Faces of the Dodecaedron into sixty Triangles, as likewise the twenty Faces of the Icosaedron into sixty. Now each of these Triangles, as ABC. is equal to the Rectangle of AD its Apotome, and of BD the Half of BC, as is also GEH

* B. 4. n. 153. † B. 4. n. 67. ‡ B. 3. n. 84.

§ B. 3. n. 54. || Sup. n. 160.

* Sup. n. 153. † Sup. n. 168. ‡ B. 4. n. 164.

§ B. 4. n. 165. || B. 4. n. 146.

GEH to the Rectangle of EF by GF the Half of GH: *
Therefore, &c.

COROLLARY. THEREFORE the Surface of the Dodecaedron is to that of the Icosaedron, as $AD \times BD$ is to $EF \times FG$. 175

LEMMA 6. THE Apotome of a Pentagon is equal to the Half of a Line equal to the Side of the Hexagon and of the Decagon inscribed in the same Circle. Eucl. 14. Prop. 1. 176

BC is the Side of a Pentagon, consequently FC, the Side of half the Arc BFC, is the Side of the Decagon, and AC the Radius of the Circle, the Side of the Hexagon. AE is the Apotome of the Pentagon. It must be proved that AE is equal to the Half of AC+CF. Make GE equal to EF; and therefore $GC=FC$. † 1. FC being the tenth Part of the Circle, the Angle FAC is 36 Degrees; and since the Arc CD contains four times 36 Degrees, the Angle CFD is then of 72 Degrees, the Double of 36. † Seeing that $GC=CF$: then FCG is Isocles, the Angle FGC is then also of 72 Degrees: CGA is then of 108. § Add GAC of 36, and it will be 144 which deduct from 180, there remains 36, the Content of ACG, which is therefore equal to GAC; consequently $AG=GC=FC$: Therefore $AG=FC$. Hence $AG+GE=FC+FE$, or $AE=FC+FE$: Therefore $AE+EF+FC$ is the Double of AE: Then AE is the Half of $AE+EF$, or the Radius $AF+FC$; and consequently half the Side of the Hexagon, and of EC the Side of the Decagon.

LEMMA. 7. ABC is an equilateral Triangle, AE a Perpendicular which bisects BC; I say that $DE=EF$. 177

BF is the Side of an Hexagon, therefore equal to the Radius BD; therefore $DE=EF$. ||

THEOR. 14. THE Surface of a Dodecaedron is to that of an Icosaedron, as the Side of the Cube is to the Side of the Icosaedron, both being inscribed in one and the same Sphere. Eucl. 14. Prop. 5. 178

AD is the Side of the Triangle which is one of the Sides or Faces of the Icosaedron, and BD is the Side of one of the Faces of a Dodecaedron inscribed in the same Circle. * EF equally divides AD, as EC doth BD; and consequently the Arc BCD: † therefore CD is the Chord of the Decagon. H is the Side a Cube intcribable in the same Sphere. 1. $\therefore EC+CD$. EC. CD. † Now EG is the Half of $EC+CD$, § and EF the Half of EC, || as is also $EG-EF$ the Half of CD: For $2EG=EC+CD$

* B. 2. n. 135. † B. 1. n. 58. † B. 2. n. 59.

§ B. 2. n. 17. || B. 1. n. 61.

* Sup. n. 172. † B. 1. n. 92. † B. 4. n. 154.

§ Sup. n. 176. || Sup. n. 177.

+CD, * and $2EF = EC$: † Then $2EG = 2EF + CD$. Taking away $2EF$ from each side, it becomes $2EG - 2EF = CD$; then $EG - EF$ is the half of CD . It has been shewn that $\frac{EC}{+CD} = \frac{EC}{CD}$: Then their Halves are in the same Ratio; therefore $\frac{EG}{EF} = \frac{EG - EF}{EF}$. But the Line H being the Side of a Cube, if it be divided into mean and extreme Ratio, its greatest Segment will be BD : † Hence $\frac{H}{BD} = \frac{H - BD}{BD}$. Then $H : BD :: EG : EF$. ‡ Then $H \times EF = BD \times EG$. Now $H : AD :: H \times EF : AD \times EF$. ¶ Then $H : AD :: BD \times EG : AD \times EF$; that is, as H , the Side of the Cube, is to AD , the Side of the Icosaedron, so $BD \times EG$ is to $AD \times EF$; but these two Products are to each other in the same Proportion as the Surfaces of the Dodecaedron, and Icosaedron: * Therefore, &c.

Q. E. D.

179 THEOR. 15. THE Radius AB being divided into mean and extreme Ratio at C , and the greater Segment AC , the Square of BG the Side of the Cube, is to that of BK the Side of the Icosaedron,

as $AB + AC$ is to $AB + BC$. Eucl. 14. Prop. 6.

LET $BFGHI$ be a Pentagon and one of the Faces of the Dodecaedron, and BKL a triangular Face of the Icosaedron, (which they may be.) † BG is the Side of a Cube inscribed in one and

the same Sphere: † Then $BK = \frac{2}{3}AB$. § Now $AB + CB =$

$\frac{3}{2}AC$: * Then is BK triple AB , as $AB + CB$ is triple AC .

Hence $BK : AB :: AB + CB : AC$ Permutatively, $BK : AB + CB ::$

$AB : AC$. Now BF is the greater Segment of BG , (the Side

of the Cube divided into mean and extreme Ratio;) † then $AB :$

$AC :: BG : BF$, † $:: BK : AB \times CB$. § Permutatively, $BG : BK :: BF :$

$AB + CB$. Now $BF = AB + AC$. ¶ and AB and AC being

Sides of the Hexagon and of the Decagon; * $AB + AC$ be-

ing put in the Place of BF , we shall have $BG : BK :: AB + AC :$

180 $AB \times CB$; which was to be proved.

THEOR.

* Sup. n. 176. † Sup. n. 177. ‡ Sup. n. 160. § B. 4. n. 153.
§ B. 4. n. 67. ¶ B. 4. n. 75.

* Sup. n. 175. † Sup. n. 172. ‡ Sup. n. 160.
§ B. 4. n. 146. ¶ B. 4. n. 70.

† Sup. n. 160. ‡ B. 4. n. 67. § B. 3. n. 53.
¶ B. 4. n. 164. * B. 4. n. 151.

THEOR. 16. *As the Side of the Cube, is to the Side of the Icosaedron, so is the Dodecaedron to the Icosaedron both being inscribed in the same Sphere.* Eucl. 14. Prop. 7. 180

CONCEIVE right Lines to be drawn from all the Angles of the Dodecaedron, and likewise of the Icosaedron, to the Center of the Sphere: which will make in one, twelve Pyramids, and in the other twenty, which are of the same Height, seeing that the same Circle of the Sphere contains one of the Faces both of the Dodecaedron and of the Icosaedron; * therefore these Pyramids are as their Bases, † that is, as the Surfaces of these two Polyedrons. Now these Surfaces are one to another ‡ as the Side of the Cube is to the Side of the Icosaedron.

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WHAT follows will be easily perceived, if what is here mentioned, be actually done upon the Sides or Faces of those Solids; whose true Forms are exhibited in Paste-board, Plate 1.

PROB. 7. *To inscribe a Tetraedron in a Cube.* Eucl. 15. Prop. 1. 181

FROM A one of the Angles of the Cube X, imagine the Diagonals AB, AC, AD, and BD, BC, CD to be drawn, and you will perceive a Tetraedron ABCD or Pyramid composed of four equilateral Triangles, for the Diagonals are equal; therefore the Triangle ABC=BAD, &c.

PROB. 8. *To inscribe an Octoedron in a Pyramid or Tetraedron.* Eucl. 15. Prop. 2. 182

DIVIDE all the six Sides of the Pyramid or Tetraedron equally in two; join the Points of Section by twelve Lines which are all equal, and make eight equilateral Triangles.

PROB. 9. *To inscribe an Octoedron in a Cube.* Eucl. 15. Prop. 3. 183

HAVING found the Center of each Side of the Cube, join them by drawing as many Lines as are requisite, viz. twelve which being equal, they make eight equal Triangles, and consequently an Octoedron.

PROB. 10. *To inscribe a Cube in an Octoedron.* Eucl. 15. Prop. 4. 184

1. DIVIDE all the Sides of the Octoedron into two equal Parts. 2. Draw Lines thro' all those Sections upon the Sides of the Body, these Lines are all equal, and make two opposite Squares, which being joined by four other Lines, which are equal to the first Lines, and the opposite ones being Parallel, they make a Cube; which is evident.

A 2 2

PROB.

* Sup. n. 172.

† Sup. n. 123.

‡ Sup. n. 198.

185 PROB. 11. To inscribe a Dodecaedron in an Icosaedron.
Eucl. 13. Prop. 5.

FIVE of the Icosaedron's Triangles which make a solid Angle or Pyramid, have for their base a Pentagon. § Divide therefore all the Sides of this Pyramid equally in two; the Plane of this Section will be a Pentagon and one of the Faces of the Dodecaedron; for by doing the same to all the solid Angles or Pyramids of the Icosaedron, which are 12 in number, there will be twelve equal Pentagons, which constitute the twelve Faces of the Dodecaedron.

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THE Diameter of the Sphere being supposed 1000, the Sides of the regular Bodies inscribed therein, are nearly as the following Numbers.

The Tetraedron	_____	_____	816
The Octaedron	_____	_____	707
The Hexaedron or Cube	_____	_____	577
The Icosaedron	_____	_____	527
The Dodecaedron	_____	_____	357

Recreative Problems.

186 PROBLEM. 1. To prolong a Line which is too short, Geometrically.

ALTHOUGH in Euclid this be taken for a Principle, yet in practice when the given Line is very Short, it will be difficult to perform it exactly, by the bare Application of a Ruler. Let the given Line be AB, with the Extent thereof and one foot in A, describe thro' B the Arc CBD, from which take equal Arcs, as BC, BD, from C and D with an equal Extent describe two Arcs, and the Point where they intersect will be in a right Line with AB, thus may the Line given be prolonged; as is evident.

187 WHEN there is a Wall as CD, or any other Impediment at the End of the given Line, proceed thus.

CD is a Wall at the End of the given Line AB, required to be prolonged from its End B. From B let fall a Perpendicular BE, at right Angles with AB, and of any length you please, draw Lines from the End thereof to any two Points in AB, as EF and EA, then make the Angle BEG equal to BEF. and the Line EG equal to the Line EF; also make the Angle BEH equal to the Angle BEA, and the Line EH equal to EA: then will a Line drawn thro'

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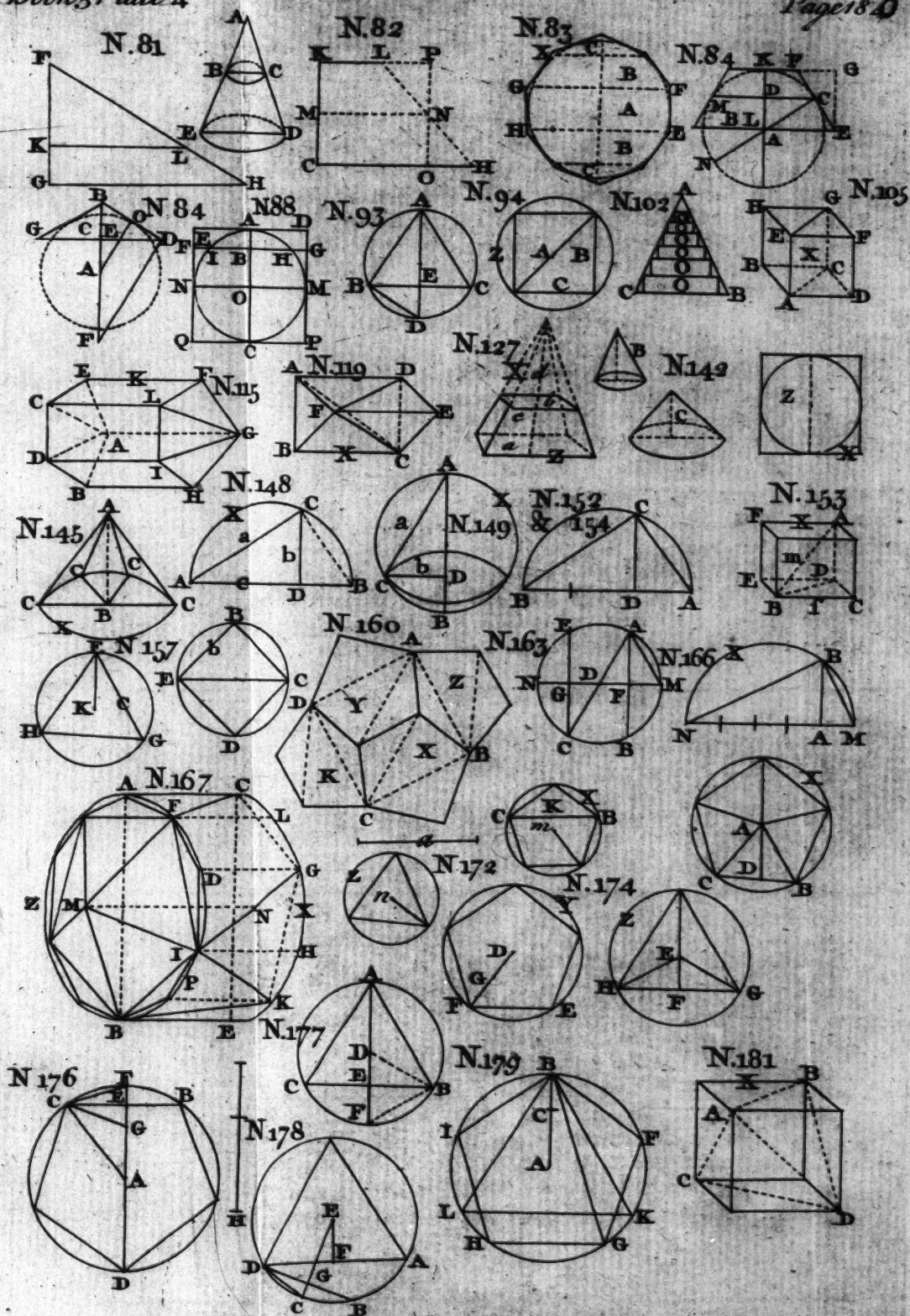
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thro' G and H be in a right Line with AB; which was required.

PROB. 2. To transform a given Triangle, into another equal Triangle, having each Side thereof longer than each Side of the Triangle given. 182

LET the given Triangle be ABC, prolong its Base AB on both Sides to D and E, making AD equal to AC, and BE equal to BC; then draw at right Angles to DE, a Line equal to the fourth Proportional between DE, and AB, the Base of the given Triangle, and CH the Height thereof, such is EG or DF, and compleat the Parallelogram DEGF, then draw Lines from F and G, to any point of the Line DE which is between A and B, and they will form the Triangle required.

SCHOLIUM. If the Lines drawn from F and G do not reach to the Line DE, but join at some Point within the Parallelogram, and under the Line AB, they will form a Triangle whose Content is lesser than that of the given one, and all its Sides longer.

PROB. 3. To divide the Circumference of a given Semi-Circle, into two such unequal parts, that the Semi-Diameter, or Radius, may be a mean Proportional between the Chords of those two Arcs. 189

LET the given Semi-circle be ABC, its Center D. From B the End of the Diameter AB, describe thro' D the Arc DE, and having divided the Arc BE into two equal parts at C, draw the Chords AC, BC, between which the Semi-Diameter AD or CD is a mean Proportional.

PROB. 4. To describe two different Isosceles Triangles, having each the same Area, and the same Circumference. 190

Draw AB, equal to 24 on a Scale of equal parts, and DE=40, upon the Middle of each draw the Perpendiculars CG=35, and FH=21, draw CA, CB and ED, FE, then will the Area and Circumference of the Triangle ACB be each equal to those of the Triangle DFE.

PROB. 5. To describe three different right angled Triangles, having equal Areas. 191

Draw for the Base of the First the Line AB=42, and the Perpendicular AC=40; And draw the Hypotenuse BC which is equal to 58 on the same Scale. Then draw DE=70, for the Base of the Second, and the Perpendicular DF=24; then will the Hypotenuse FE=74. Lastly draw FG=112, and the Perpendicular EH=15. Then is the Hypotenuse GH=113. Then proceeding with those numbers according to Book 2. n. 135. It will appear that the Areas of the Triangles CAB, FDE and HFG are equal one to another.

PROB.

- 192 PROB. 6. To describe three equal Triangles, the First of which is right Angled, the Second an Oxygonium, and the Third an Amblygonium.

Draw $AB=24$, for the Base of the right angled Triangle, and $AC=7$, then is the Hypothenufe $BC=25$. Secondly, For the Base of the Second draw DE and take therefrom the Segment $KD=8$, and $KE=9$, and from the Point K raise the Perpendicular $KF=12$; and compleat the Triangle by drawing the Lines FD and FE . Thirdly, Draw GH for the Base of the Obtuse angled Triangle, and make the Segment $LG=6$, and $LH=15$, Erect the Perpendicular LI at the Point L equal to 8; and draw the Lines IG and IH which form the Triangle required: then are those three different Triangles equal to each other; as was proposed.

- 193 PROB. 7. To describe a right angled Triangle, having its three Sides in Geometrical Proportion.

With any Extent describe a Semi-circle as ABC , whose Center is D , take the Diameter AC for the Hypothenufe of the Triangle required: From C , the Extremitie of AC , draw the Perpendicular CE , equal and Perpendicular to the Diameter AC , and draw DE , which cuts the Circumference at F . Take the Distance of FE and set it off upon the Circumference from A to B , and draw the Lines AB , BC , then is ABC right angled at B , and its Sides are in the required Ratio.

- 194 PROB. 8. To describe a right angled Triangle, having its three Sides in Arithmetical Proportion.

DRAW an indefinite Line and mark thereupon five equal parts, as here from A to B ; and account AB the Hypothenufe of the Triangle required. From A , with the Extent of three of those equal parts, draw an Arc, and from B , with the Extent of four parts, draw another Arc, which will intersect the First at C , draw the Lines AC and BC ; and they will form the Triangle required.

- 195 PROB. 9. Two Lines being given Perpendicular to a Line drawn thro' their Extremities, to find upon that Line a Point equally distant from each of the two other Extremities. Or, the Breadth of a Street being given, to find in what part thereof a Ladder of a determinate Length must be placed, so that its Top may reach from thence to two Windows of different Heights, on the opposite Sides of that same Street.

LET AB be the Breadth of the Street, BC the Height of one Window, and AD the Height of the other, join D and C by the right Line DEC , from E the Middle thereof let fall the Perpendicular EF upon the Line AB , then is the Point F the Place required, the Line FC being equal to FD .

the

Suppose $AB=19$, $BC=56$, $AD=63$, then $AF=16$, $FB=33$, and FC and FD each equal to 65.

PROB. 10. To describe in a given Circle four equal Circles mutually touching each other, and likewise the Circumference of the given Circle.

LET the given Circle be $ABCD$, its Center E , draw the Diameters AC , BD Perpendicular to each other, take DE equal to CD , the Chord of Ninety, then set EF from A to F , from B to G , from C to H , and from D to I ; and then will the Points K , G , H , I , be the Centers of the Circles required.

PROB. 11. To describe four equal Circles which mutually touch each other, and also the Outside of the Circumference of a given Circle.

THE given Circle is $ABCD$, in which draw at right Angles the Diameters AC , BD , which intersect at the Center E ; prolong the Diameter AC and take AF equal to AB , the Chord of Ninety, then is EF the Radius of each of the Circles required.

PROB. 12. To describe six equal Circles which mutually touch one another, and also the three Sides, and three Angles of an equilateral Triangle given.

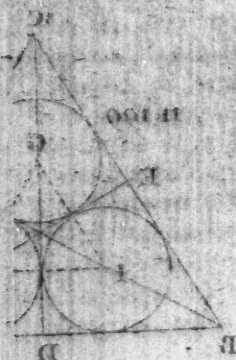
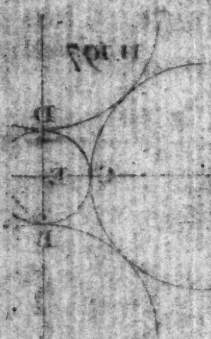
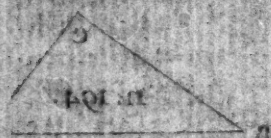
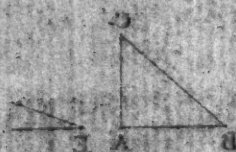
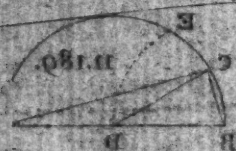
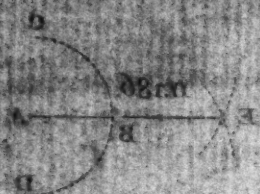
LET ABC be an equilateral Triangle given, whose Center is D , Bisect each Angle equally in two and draw the Lines AF , CE , BG , of a convenient Length. Upon the Side AB take EH equal to half the Perpendicular DE , draw DH , and prolong it making $HI=HE$, then is the whole Line DI the Radius of each Circle, which being set off upon the produced Lines from E to K , B to L , F to M , &c. will give the Centers required.

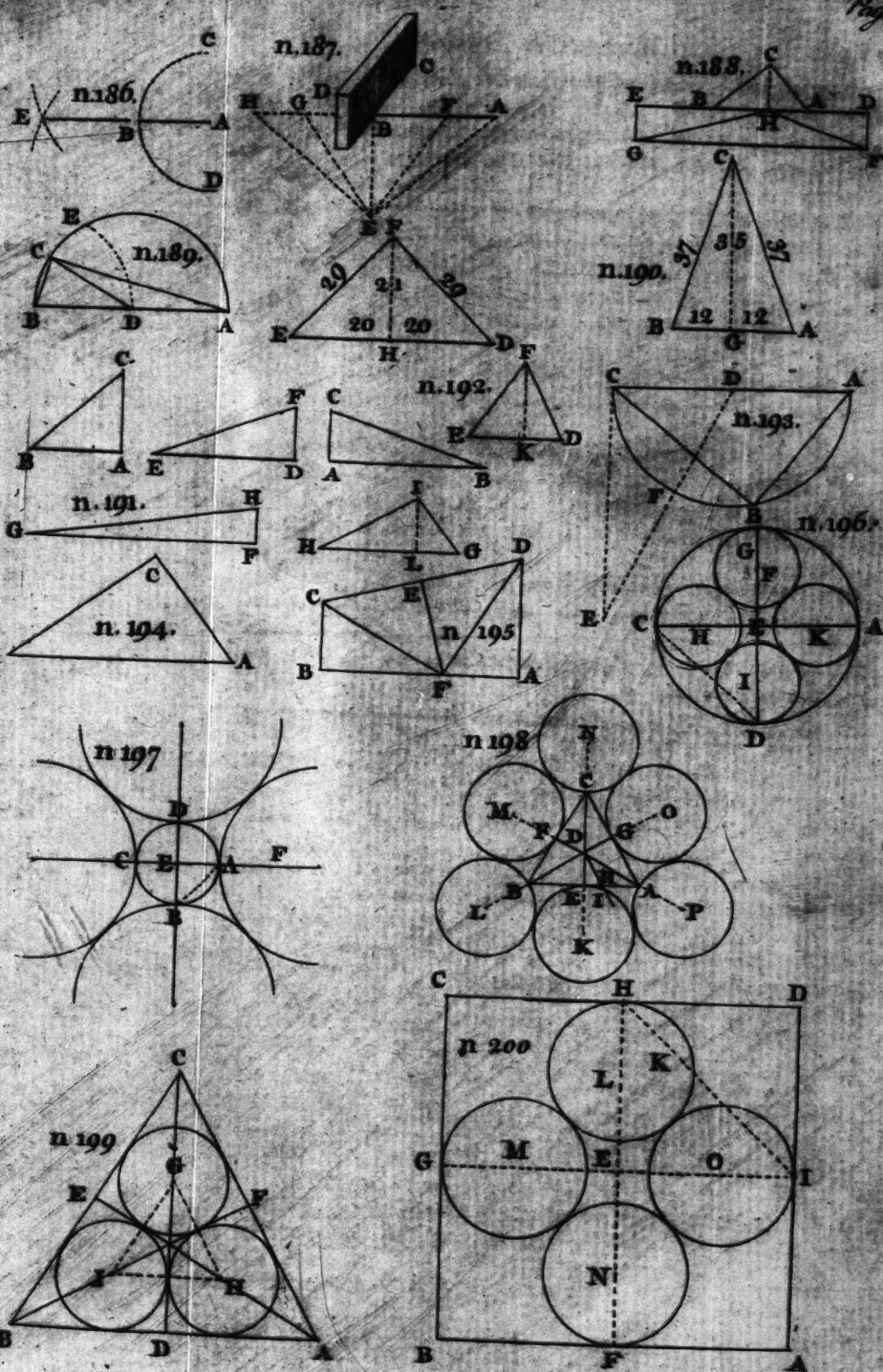
PROB. 13. To describe within an equilateral Triangle three equal Circles which touch one another, and also the three Sides of the equilateral Triangle.

LET ABC be an equilateral Triangle, divide each of its Sides into two equal parts at the Points D , E , F , and thro' these Points draw Lines to the opposite Angles, then take the Extent of AD or DB , the Half of the Side of the given Triangle, and set it upon the Line AE from E to H , and also upon the Line BF from F to I , and upon the Line CD from D to G , then are the Points G , H , I , the Centers of the Circles required: these Points being joined by right Lines, will form another equilateral Triangle having its Sides parallel to the Sides of the Triangle given.

PROB. 14. To inscribe in a given Square, four equal Circles which touch one another, and likewise the Sides of the Square.

Divide each Side of the Square $ABCD$, into two equal Parts at the Points F , G , H , I , and draw the Lines FH , GI , which cut one another at right Angles in two equal parts at B , the Center of the Square, join the Points H , I , and take therefrom the Part $IK=IE=EG$, the Remainder KH is the Radius of each Circle, which set off upon the Lines FH , GI , from the Extremities thereof, and it will mark out the Center Points L , M , N , O .





BOOK VI.

OF METHOD.

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THERE are two ways or methods of Solving Problems. The First, which proceeds from known or given Quantities, to the Discovery of the Thing required, has been chiefly made use of in the foregoing Part of this Treatise, and is commonly called the *Method of Doctrine*, whereas the Second, which is more Universal, begins with what is unknown and proceeding therewith as if it had been known, it is at last found equal to some known or given Quantities, and from thence its true Value is discovered: This is called the *Method of Invention*, and has been made use of heretofore, on certain occasions, in order to find what was unknown, and which we shall now proceed to apply to the Solution of several useful Geometrical Problems.

CHAPTER I.

OF the Examination of a Question, or Problem. In the first Place, it must be well understood, and exactly expressed.

The Truth often lies so concealed in Problems whose Solutions are sought, as not to be discovered by the bare application of the Mind: The Imagination must therefore be assisted by drawing such a Figure as is proper to express those things which are required to be considered. And this may be done, altho' some of those things are unknown to us, but something must be given otherwise it is impossible. Now from that which is given, we deduce such Suppositions as the Nature of the Problem requires: Then drawing Lines according to what was supposed; those Lines will be known, as will appear farther on, and from thence we discover those which are required: Therefore the first thing that ought to be done, is to form such a perfect Idea of the Question, as to be able to express and distinguish exactly those

B b

things

things which are to be found, and to free it from every Superfluity which serves only to render it obscure. The following Examples shew that difficult Questions are often solved by the bare Expression thereof. I here mean figural Expressions as well as Verbal ones.

QUESTION 1.

- 2 To Demonstrate that the Surface of a Triangle is equal to half the Sum of its three Sides, multiplied by the Radius of its inscribed Circle.

THE bare Sight of this Figure demonstrates that this is true. Having drawn Lines from the Angles of the Triangle BCD, to A the Center of the inscribed Circle, they make three Triangles BAC, CAD, DAB, together equal to the Triangle BCD, and whose height is the Radius of the Circle; they are therefore equal to the three Sides of BCD, and whose height is the Radius of the inscribed Circle. * the Surface of the Triangle is therefore equal to the Product of half its Base by its Height; which was to be proved.

ALSO see here by another Example, how the Manner of expressing a Question by a suitable Figure, facilitates the Resolution.

- 3 QUEST. 2. To demonstrate that in the Triangle ABC, if a Perpendicular be drawn from the Angle CAB upon BC, the Sum of the two Sides, AB and AC is to BC, (the Base of the Angle which those two Sides contain,) as the Difference between CD and BD is to that between AC and AB.

FROM A as a Center, and with the Extent of AB, the shortest Side describe a Circle, and since AB is equal to AE, the Line CE is the Sum of the Sides AC and AB. The Lines AF and AB are equal; therefore CF is their difference. Also seeing DB=DG, the Line GC is the Difference between CD and DB. Hence here is an expression, or Figure which discovers what is sought. Therefore the Question is now easy to be solved; for $CE \times CF = BC \times CG$; † Therefore CE. CB::CG. CF; ‡ Q. E. D.

C H A P. II.

- 4 ALTHO' the Lines and Quantities concerned in a Question are unknown, yet they may be expressed, and all the Operations of Arithmetic performed upon them.

The

* B. 2. n. 143.

† B. 4. n. 57.

‡ B. 3. n. 58.

THE Method which we now treat of is Universal, and very different from that which hath been used hitherto. The Business is here, to find and demonstrate the Properties of the Figure which is in Question; and in this Pursuit, we consider Quantities only as being to be added, Subtracted, Multiplied, or divided. And how this may be done upon every thing that is capable of being augmented or diminished, whether Lines or Numbers is shewn in Book the Third: hence this Method is already partly explained; therefore if the preceeding Books be read, we cannot then be ignorant that by $a+b$, may be understood two Lines a and b added together; by $a-b$, that b is subtracted from a ; by ab that a hath been multiplied by b , or b by a : that therefore ab is a Rectangle, whose Sides are a and b ;

that aa , or a , is a Square, one of whose Sides is a ; that $\frac{ab}{b}$ is a Rectangle, or Plane divided by b , after which Division there remains only the Side a , that is, a Line only. According to what was taught at the beginning of the third Book, the Letter which is both above and below the Sign of Division may be

cancelled; and that therefore $\frac{ab}{b} = a$. Powers may likewise

be expressed by Letters as well as by Lines; the Proportions of Figures may therefore be denoted by Letters; for if a Triangle is right angled, and its Hypothenuse be a , and the two Sides b and c , according to what has been taught, * $aa=bb+cc$, and $aa-bb=cc$, and $aa-cc=bb$.

The Proportions of similar Triangles may likewise be all expressed by Letters; for if ABC and DEF are similar, that AB $=a$, and BC $=b$, and DE $=m$, and EF $=n$; then $a::b::m::n$. and seeing that according to what has been said, b multiplied by m , and the Product bm divided by the first Term a , the Quo-

tient $\frac{bm}{a}$ will be the fourth Term: Then EF $= \frac{bm}{a}$, and by the

same Reasoning $\frac{bm}{a} = AB$. Which shews how to denote the

Proportion of Lines, as well as those of Numbers. This universal Method of expressing Quantity, and performing Arithmetical Operations thereupon, is called *Algebra*, which, tho' it be difficult to those who have not been accustomed thereto, it will not be so to such as make use of these Elements. Observe that the Product of a Multiplication may be reduced to a Proportion; for Example, the Product of a by b , to a Proportion whose first Term is unit, the Second and Third are the two Quantities a and b , which multiply together; and the fourth Term will be ab , which is the Product of the Multiplication, which

* B. 2. n. 142.

† B. 3. n. 60.

which is evident, For let the fourth Term be called x : Then

1. $a::b. x$. Now $\frac{ab}{x} = x$. * But Unit neither divides nor

multiplies, therefore $\frac{ab}{x} = ab$, and therefore $ab=x$. Unit

may therefore be supposed tho' it be not expressed. Hence the Product of three Letters, as abc , may denote these two Proportions . 1. $c::ab. abc$; and 1. $ab::c. abc$. This shews that the Product of four Lines $abcd$, may denote three Proportions, the Product of five Lines $abcde$, denotes four Proportions; and in these Proportions the whole Product is only a Line, which proceeds from all these Proportions, when the first Term is Unity; for seeing that 1. $a::b. ab$; then 1. $bua. ab$ likewise, since 1. $ab::c. abc$; then 1. $c::ab. abc$. The Quotient of Division

expresses a Line, for Example, $\frac{a}{b}$: For it may be reduced to

this Proportion $b. a::1. x$. for $\frac{a1}{b}$ or $\frac{a}{b} = x$.

8 THE Operations of Arithmetic are all easily performed by the Ruler and Compasses, one Line may be added to another, or a Lesser be subtracted from a Greater. To multiply one Line by another, *that is*, to find a Line equal to the Product of two Lines; proceed thus. Let the Lines given to be Multiplied be AD and AC; upon AC take the Line AB equal to Unit, and draw a Line AD making any Angle with AB. Draw a Line thro' B and D, and thro' C another Parallel thereto, which being done, AE is the Line required: For AB. AD::AC. AE; then $AB \times AE = AD \times AC$. A Quantity being multiplied by only Unit continues the same; therefore AE multiplied by AB, which is a Unit; continues equal to the same as before, therefore AE, which is a fourth Proportional, is equal to the Product of AD by AC; *which was sought*. If it be required to divide AE by AC, having taken AB equal to Unit, and drawn thro' B, a Parallel to CE, the Line AD is the Quotient of AE divided by AC; for AB. AD::AC. AE, and Unit is to the Quotient of a Division, as the Divisor is to the Dividend. If

10 it were required to extract the square Root of GH, add thereto a right Line FG, equal to a Unit, and divide FH into two equal parts at the Point E: From the Center E describe the Circle FGH. Erect a Perpendicular from the Point G, and the Line GI, is the Root required; for $\frac{FG}{GI} \cdot GI \cdot GH$. Then the Square of GI, is equal to the Product of GH. Now FG being Unit, if GH be multiplied thereby, it will yet continue the same, therefore GH is equal to the Square of GI, which is consequently the Root of GH, the same Way may be found a Line equal

equal to the square Root of a surd Number; for Example, equal to the square Root of 18: for taking GH equal to 18, and adding thereto FG equal to unit; and from E the Middle of that Line as a Center, describe a Circle, then is the Line GI equal to the square Root of 18, which as we have shewn in the fourth Book, is not expressible by any number.

C H A P. III.

AFTER having expressed a Question or Problem; and described the Figure proper thereto, it must be considered whether the Problem be determinate or indeterminate; and that which is known must be distinguished from what is unknown.

THE Problem being exactly conceived, and the Figure described which expresses its conditions, there must be distinguished these three things. 1. The known Lines. 2. The unknown ones. 3. All the Proportions which the known and unknown Lines bear one to another. Some of the Lines which are drawn to express a Problem are known, the Rest are supposed, *that is*, 'tis supposed that they ought to have such and such Proportions with the known Lines. We shall here represent the known Lines by the first Letters of the Alphabet, calling them *a*, or *b*, or *c*, &c. and the unknown ones by the Last, *x*, *y*, *z*. It may sometimes be proper to use the first Letter of the Names of the known and unknown Quantities, for Example, to represent a number by *n*, a Sum by *s*, time by *t*, velocity by *v*. Next consider whether the Problem be so proposed, as that the drawing certain Lines, depends upon the Choice of certain Quantities at pleasure, or whether all those which the Question contains, are determined, so that all which can be done, be only to give them suitable names. It must then be immediately examined whether the Question be determined, or indetermined. In an indetermined Question, there appears no Proportion, which gives room to express the unknown Quantities two ways, for if I can call the same Quantity either *x*, or *y* + *a*. 'tis because I know that *x* is equal to *y*, after *a* is added thereto; that $x = y + a$, this is a determined Proportion. But when, without determining any, it is for Example proposed to divide the Diameter AB, so that the Rectangle of the two Parts may be equal to a Square, the Problem is indetermined: and I cannot find a double Expression, but by supposing a certain Proportion; as this for Example, that AB is divided in C. After which, having described a Circle with the Extent of half AB, and upon C raised the Perpendicular CD, I shall have what was required.

$\therefore$ AG. CD. CB: * Then $AC \times CB = CD^2$. † An indeterminate Problem admits of several Solutions; for what part soever of the Diameter C is in, the Square of CD will be equal to the Rectangle of the Parts of the Diameter if divided as supposed to be. AB may be divided in many places besides C.

- 13 A Problem is said to be determinate, when it cannot be Solved but by observing certain things which limit it, and which depend not upon the Will of the Proposer, *that is*, the Quantities contained therein, have a certain Proportion peculiar to themselves. For Example, AB being divided in C; it is proposed to produce AB even to D, (*an unknown point*;) so that the Square of CD may be equal to the Rectangle of AD and BD: then as the prolonged part BD is determinate, *that is*, that it hath a certain exact Length; the Problem is determinate: And suppose that $AC = a$, and $CB = b$, and $BD = x$, there will be this double expression, *viz.* For the Rectangle, $ax + bx + xx$, and for the Square $bb + 2xb + xx$; which two Quantities ought (*according to the Supposition*) to be equal; therefore $ax + bx + xx = bb + 2bx + xx$, hence may the value of x be discovered.

C H A P. IV.

- 14 **T**HE Proportions which are between the Lines of the Figure by which a Problem is expressed, being known, thence arises the Method of equalizing them, or making double expressions thereof, which is called an Equation.

If it be known that three given Lines a , b , c are in Proportion, and that the unknown x is the fourth Term of that Proportion; that therefore $a. b :: c. x$, † then seeing that $\frac{bc}{a} = x$,

there is this double expression of the same Quantity, $\frac{bc}{a}$ and x ,

and this is what is called an Equation. To find Equations, is to find double Expressions of an unknown Quantity; which may be done, when some of its Proportions with the given Quantities are known. If it is known that the Sum of a and b is equal to the unknown Quantity x , this Expression $a + b = x$ is an Equation. It is evident that the Half of two unequal Lines, less half their Difference, is equal to the least Line; and that same half, more half their difference, is equal to the greatest Line.

LET the Line $AC = x + z$, $AD = x$, and $DC = z$, let their difference be DE, whose half is BD. Let $AD = y$, and $DE = a$, hence

* B. 4. n. 29.

† B. 3. n. 57.

‡ B. 3. n. 60.

hence DB or $BE = a$, it is plain that $AD - DB$, or $y - a = AD$ or x , and that $BC + BD$, or $y + a = DC$ or z . Hence when the Difference between two unknown Quantities is known, they may be expressed two different ways.

THE Proportions which are between the Lines of a Geometrical Problem, are often discovered by means of the Figure itself, which the Problem is expressed by; for if for Example, I know the unknown Line x , to be the Hypotenuse of a right angled Triangle, whose two other Sides are d and b , I know that $xx = bb + dd$; * hence here is a double Expression of x . A Plane is produced by the Multiplication of its two Roots; if it be therefore divided by one of them, the Quotient of the Division is the other. Thus if $b :: d :: x$, since that $bx = ad$, dividing cd by b , the Quotient $\frac{cd}{b} = x$. If x were a mean Pro-

portional between c and d , then $xx = cd$. That which was delivered in the third Book concerning Powers, is also a Fund of different Equations: For if I know that $a + b = x$, then $aa + 2ab + bb = xx$. † If x is the Hypotenuse of a Triangle, and that the two Sides are a and b , then $aa + bb = xx$: wherefore $xx - aa = bb$, or $xx - bb = aa$. The same Quantities may be differently expressed, by denoting either the Parts for the Whole, or the Whole for the Parts, the Powers for the Roots, or the Roots for the Powers, for if $a + b = x$, then $aa + 2ab + bb = xx$; or $aa + 2ab = xx - bb$, or $aa + bb = xx - 2ab$. It is evident that an infinite number of Equations may be found this way: if $ab = xx$; then $\sqrt{ab} = x$.

CHAP. V.

THERE must be found as many Equations as there are unknown Lines, and all these Equations must be reduced to one only.

THE main Scope or Design in a Question, or Problem, is to bring back, (If I may so say) the unknown Quantity, and to compare it with the known, so as to observe their Difference, thence is discovered what must be added or subtracted, in order to render them equal; which may also be done when their Ratio is exactly known: For if x is the third Part of a , then $a = 3x$. If 'tis a which is third Part of x , then $3a = x$. Thus may Equations or double Expressions be found for all the Quantities in the Problem; and there must be as many found as there are

* B. 4. n. 79.

† B. 3. n. 20.

are unknown Quantities. There is always in a Problem one principal unknown Line, which the Solving of the Problem depends upon: and 'tis to that Equation, that all the rest must be reduced, so that there may be only one unknown Letter; for if it is known that $x + b = z$, or that $z - b = x$, wherefore where those two unknown Quantities are, I may always substitute $x + b$ in the Room of z , or $z - b$ in the Place of x , and consequently reduce the Question to other Terms having only one unknown Quantity. *There are different Methods for doing this.* When the unknown Quantity is known to be the third or fourth Term of a Proportion, it may be expressed by the Letters of the known Quantities; for $\div a. b. x;$ then $\frac{bb}{a} = x.$ * If $a. b::c. x;$

then $\frac{bc}{a} = x.$ The Terms of a Progression may expressed so as to make use only of the same Letter. If this progression of four Terms be expressed by four different Letters $\div b. x. y.$ It may be reduced so that there remain but one unknown Letter; for suppose $b = 1$, I may then say that $\div 1. x::xx.xxx.$ 1. b or 1 is to x , as the Square of b or 1 is to that of x , † that is, $b. x::bb. xx$, or 1. $x::1. xx$; hence since $xx = x$, I may substitute xx in the Place of x . So likewise b or 1 is to y , as bbb or 1 is to xxx , † then as $xxx = y$, instead of y , I put xxx ; thus I reduce the four Quantities b, x, x, y to these 1, x, xx, xxx , there are many other such like methods of exterminating unknown Quantities.

C H A P. VI.

22 **T**HE Terms of an Equation must be reduced to the simplest Expression possible, and in such Sort that the unknown Quantity be found only in one of the Members of the Equation.

WHEN you have reduced the Equation so that the unknown Quantity is found only in one of the Members thereof, that is, on one side of the Sign of Equality, and there are only known Quantities on the other Side; the unknown Quantity is then found exactly equal to known Quantities, and the Question is entirely answered. If $x = a + b$, we can't but know the value of x . Now in order to do this, you must transpose that which is known on one Side, and free it from that which is unknown. The first thing should be to reduce the Equations to the most simple Expressions. Which is done either by Addition, Subtraction,

* B. 3. n. 60.

† B. 3. n. 70.

‡ B. 3. n. 87.

traction, Multiplication, Division, or in short, by the Extraction of Roots. The fundamental Principle of all these is that if *equal Quantities are added to, or subtracted from, both Members of the Equation, the same Equality will continue*, as it will also if both Members are multiplied, or divided by the same Quantity, as was proved. * Seeing that two equal powers have equal Ratios, it is plain that in taking the Roots of both Members, the same Equality will subsist.

If $x-5=15$, add 5 to each Side, and there will arise this simpler Equation $x=20$. If on the contrary $x+5=20$, by subtracting 5 from each side, it will be $x=15$. If $x-a=b-a$, adding a to each part, it becomes $x=b$. So likewise if $x+a=b+a$, subtracting a from each part, it becomes $x=b$. In this Equation

$\frac{x}{3} = b$, each Member multiplied by 3, it is $x = 3b$, as

on the Contrary, if it had been $bx=3b$, each Member divided by b , it would be $x=3$. If $xx=25$, extracting the Square Root, it becomes $x=5$. On the other hand, if it had been $\sqrt{x}=5$, that is, if the Square Root of x is equal to 5, by involving both members to one and the same Power, or by taking their Square, there is this Equation $x=25$. If it had been $xx=ab$, it might be put $x=\sqrt{ab}$.

THAT the Square of the Hypotenuse is equal to the Squares of both the other Sides, is a Theorem which shews how to reduce an Equation to a simpler Expression: For having for Example, this Expression $aa-bb$, and it be required to find a Square equal to $aa-bb$, take the Half of a , and from the Middle or with the Extent of that half, as Radius, describe a Circle; or having taken a Chord equal to b , a Line being drawn from its End to the other End of a , which is the Diameter of the Circle, it will form a right angled Triangle whose Hypotenuse is a , and one of its Sides b : The third Side which is known I call c ; then $aa=cc+bb$: Taking away from each Part bb , it becomes $aa-bb=cc$; hence cc may be put in the Room of $aa-bb$, and consequently we have here a simpler Expression. So if I had $cc+bb$. Join c and b together so as to make a right Angle, and compleat the Triangle, the third Side is the contained Hypotenuse: this Hypotenuse being called a , it is then evident that $aa=cc+bb$, therefore aa , which is a simpler Expression, may be Substituted instead of $cc+bb$.

To find a Square equal to a given Plane. Suppose the two Sides or Roots of the Plane are b and d . Let c be a mean Proportional between them; then $bd=cc$. Thus may the Plane bd

C c

b d

be transformed into the Square cc , that is, one may be put for the other.

- A Square being given, another may be found bearing any required Proportion thereto, either the Half, Third, Fourth, or Fifth, &c. Let the Square be aa , to find another equal to the fourth Part thereof. Take a Line at venture, upon which mark five equal Parts with the same Extent of the Compasses. Let this Line be BC , whose fifth Part is BD ; hence DC is equal to four of these Parts. From A the Middle of BC , and with the Extent of AB or AC , draw a Circle; and upon D erect the Perpendicular DE , which terminates at the Circumference of the Circle. Then $\therefore BD. DE. DC$. * If BD is called b , then $DC = 4b$. And if $DE = x$, then $\therefore b. x. 4b$; therefore $xx = 4bb$. Upon C raise a Perpendicular CF equal to a the Root or Side of the Square aa , the Square of whose fourth Part is required. Draw FG parallel to BD ; and thro' the Point G , where it intersects the Radius produced *ad. inf.* let fall a Perpendicular upon BC produced infinitely on each Side. The Perpendicular GH is equal to CF , and consequently to a . † Draw a Circle with the Extent of AG , whose Diameter is IL . It is evident that IH is the same Part of IL as BD is of BC ; consequently a fifth Part. Hence if IH be called m , the Remainder of the Diameter HL is $4m$. Now $\therefore IH. HG. HL$, or $\therefore m. a. 4m$. ‡ Then $aa = 4mm$. § The Line IH is consequently the Side or Root of a Square, equal to the fourth Part of the Square a . This Example shews how to find a
- 27 Square that shall be such a Part of a given Square as is required, which is of great use for reducing an Equation to a simple Expression. There are innumerable Ways of reducing
- 28 compound Expressions to others more Simple and Elegant: After which there is nothing more to do but to free the Equation from certain superfluous Quantities, which may be done by adding that Equation to another, which contains the same Quantities with a contrary Sign, according to this principle, *that more and less the same Quantity, is equal to nothing*. Thus, if $x = d - b$ and $x = d + b$; in order to cancel b , add the two Equations into one $x + x = 2d$, in which b is expelled. It is easy to transpose a Quantity from one of the Members of an Equation to the other; for in this $x - b = d$; adding b to each Side, it will be $x = d + b$, in which b is moved to the second Member. Altho' the Roots of an
- 29 Equation be unknown, *that is*, the Quantities of whose product the Equation is composed, yet it may be encreased or lessened as is deemed necessary. Thus if $xx = xd + bb$ whose Roots are $x, x + d, b$, and that it be required to encrease x by 3. Take the Quantity y and suppose it equal to $x + 3$. Then $y - 3 = x$, then in every

* B. 4. n. 29. † B. 1. n. 64. ‡ B. 4. n. 29.
§ B. 3. n. 59.

every place where x is, put $y-3$; instead of the Square or Cube &c. of x , put the Square or Cube &c. of $y-3$; after which the Equation $xx=xd+bb$, is transformed into this, $yy-6y+9=yd-3d+bb$, which is the same. In augmenting or diminishing an Equation, it must be so managed as to have contrary Signs; and that the Quantities may therefore be cancelled which the Equation wants to be freed of.

C H A P. VII.

EQUATIONS are of one or more Dimensions or Degrees; and by ³⁰ those Degrees it is that Problems are distinguished.

THE unknown Quantities are involved, or raised by Degrees, according to the Nature of the Problem. When the unknown Quantity which all the rest are reduced to, is not multiplied, as in this Equation $x=d+c$, the Equation is said to be simple or of one Dimension. We have shewn that when there are several unknown Quantities, they are to be all reduced to only one. If for Example there was x and z , and their difference was known to be b ; if x be the lesser, then $z-b=x$; hence $x+b$ may be put instead of z . Consequently; if by the proposing of the Question, it is known that the Rectangle of x and z , or of x and $x+b$, is equal to the Square of c , there will be this Equation $xx+xb=cc$, which by taking away xb from each side, is reduced to this $xx=cc-xb$, in which x is raised to the second Degree. If the Question had contained three unknown Quantities, and that they had been multiplied one by the other; x , which all the rest would be reduced to, would be raised to, the third Degree. Hence it appears that Equations are of one or more Dimensions, or of several Degrees. I shall say no more here of Equations which are of above two Degrees, because they cannot be solved with the Ruler and Compass, *that is*, by making use of only the Circle and right Line.

EQUATIONS of two Dimensions are reduced to these three forms, 1. $x=aa-xd$. 2. $xx=aa+xd$. 3. $xx=xd-aa$. For ³¹ if $xx=ab-d$, as ab is a known Quantity, by taking $ab=cc$, * $xx=cc-d$. If $xx=xd+c$. I can find a Square bb equal to the Quantity c , so that $xx=bb+xd$. Find a mean Proportional between Unit, and c , which being called b , then is $bb=1c$, or $bb=c$. Carefully endeavour to compare the unknown Quantities ³² so as may most tend to the Formation of the simplest Equations. By finding and reducing Equations we obtain the Solutions of Problems; now according as these Equations are reduced, Problems are distinguished. When the Equation is so reduced,

as

* Sup. n. 25.

as that it hath only one dimension, as if $x=b$, or $x=\frac{ab}{c}$, the Equation is of the first Degree, and the Problem is a simple one. When the unknown Quantity which the Equation is reduced to, hath two Dimensions as $xx=aa+bb$, which is an Equation of the second Degree, the Problem is called a *Plane*. When the Equation contains the unknown Quantity raised to the third, or fourth Degree, as $x^3=aab$, or $x^4=acb$, which are Equations of the third, and fourth Degree, it is called a *Solid* Problem. When the unknown Quantity is raised higher than the fourth Power, the Problem was by the Ancients called *Lineal*; but it is more natural to call it by the Degree which the unknown Quantity is raised to.

C H A P. VIII.

34 **O**F the Geometrical Construction or Effect of Equations, that is, of the Method of expressing by Lines the Quantities concerned therein.

THE Method of expressing by Lines the Quantities belonging to an Equation, is called the Geometrical Construction or Effect of an Equation. The Construction of Equations of one Degree is very easy, there being nothing required but to make a Line equal to a known Quantity. Thus if $x=a$, and that a be equal to a foot, it is easy with the Ruler and compass to draw a Line equal thereto. When the known Member of an Equation hath several Quantities as $x=a+b$, it is also easy, for the Equation may be reduced to a simpler Expression, by taking a Quantity equal to both the Quantities. Two different Lines may be taken and joined together. Sometimes instead of the value of a Quantity we seek its quotient when divided by other

Quantities; for Example $x=\frac{ab}{c}$: To express this Geometrical-ly, that is, to construct or make a Figure which expresses this Equation. Observe that these Letters denote Quantities proportional one to another; for if $c. a::b. x$, the second Term a being multiplied by the Third b , and their product ab , divided by the First c , the Quotient of the Division $\frac{ab}{c}$ is x the Fourth pro-

portional; hence $x=\frac{ab}{c}$. To express this Geometrically, having taken AB equal to c , and AE equal to b , upon B, raise BC= a , and from A draw a Line thro' C; then from E, draw a Parallel to BC or to a , which I call x . These four Lines

A B

AB. BC::AE. x . are proportional; hence $\frac{BC \times AE}{AB} = x$, or $\frac{ab}{a+b} = x$, as 35

these known Quantities may be reduced to this Proportion $\frac{a+b}{a+b} = x$, or $a+b::a$, or x , the same figure expresses this Equation; three lines being given a fourth proportional may be found, which is the value of x .

When the known Quantities of an Equation have a radical Sign, they may be likewise expressed geometrically. If for example, $x = \sqrt{ab}$, by drawing a line AB equal to a , and producing it even to C, so that BC = b , and from the middle of AC, as a center, describe a circle, then is the perpendicular BE the Value of x ; for $\because$ AB. BE. BC, or $\because a. x. b$; hence $ab = xx$: Therefore $\sqrt{ab} = x$. This Equation $x = \frac{bb}{a-b}$ may be al- 37

so constructed and expressed thus. AD being taken equal to a , and AB to $a-b$, hence BD being equal to b , I raise upon B a Perpendicular equal to BD or b , draw a right line from A to E; and upon AE at the point E draw a perpendicular, viz. EC, which cuts the produced Part of AD: then is the Angle CEA a right one; having therefore described a circle thro' the three Points A, E, C, it is evident that $\because$ AB. BE. BC, or that $\because a-b. b. x$. put x for the value of BC, then bb , divided by $a-b$ is equal to x ; * hence $\frac{bb}{a-b} = x$. Equations of two Degrees are also easily 38

constructed; and are, as we have shewn, reducible to one of these three Forms $xx = ax + bb$, or $xx = ax - bb$, or $xx = bb - ax$. 1. For the Construction of $xx = ax + bb$, or $xx = bb - ax$; upon a right line CD, which I suppose equal to b , I erect a perpendicular CE which I suppose equal to the half of a , and with the extent of that half, as Radius, I describe a Circle, and thro' its Center E, draw the Line DE. Since that CE is the half of a , then AB = a . Let BD = y , and AD = x . then $a + y = x$; therefore $\parallel ax + y = xx$. Now seeing that CD or b is a Tangent, then $\dagger \because x. b. y$; therefore $\dagger xy = bb$. Hence substituting bb in this Equation $xx = ax + xy$ in the Room of y , which is the same, there will arise this Equation $xx = ax + bb$, which is the construction required. 2. The construction of this Equation $x = \frac{bb - xa}{a}$, is the same. CD (same Figure) is supposed equal to b , and CE equal to the Half of a ; but 'tis supposed that BD is equal to x . Then $\because$ AD (or $a + x$) CD (or b) BD (or x) then 39

* B. 3. n. 60.

|| B. 3. n. 18.

† B. 4. n. 33.

‡ B. 3. n. 57.

- $ax+xx=bb$. By taking away ax from each part, there will remain $xx=bb-ax$, which is the Equation required to be constructed. 3. This Equation $xx=bx-bb$, is constructed thus.
- 40 Let BA be equal to a , with the extent of BC, the half of AB, describe a Circle; upon B, erect the Perpendicular BD equal to b ; thro' D draw the Line DG Parallel to AB. If this parallel don't cross the Circle, which will be the Case if b is equal or greater than the Radius of the Circle, which is equal to the Half of a , 'tis a Proof that xx is not equal to $ax-bb$, since that b is too great with respect to x . Let then x be lesser than BC, hence DG intersects the Circle in E, consequently EF which I suppose perpendicular, is equal to BD. § Suppose $AF=x$, and $FB=y$. It is evident * that $\div AF$. (or x .) $\div FE$. (or b .) $\div FB$. (or y .) hence $\div x. b. y$: therefore $xy=bb$. Now AF (or x) more FB (or y), hath been supposed equal to a ; therefore $x+y=a$. This Equation multiplied by x , produces $xx+xy=ax$: Substituting bb , in the room of its Equal, xy , it is $xx+bb=ax$; and subtracting bb from each Side, it will be $xx=bx-bb$, thus is this Equation constructed. By means of this Construction the Value of the unknown Quantity is found by a Line; which is found yet more readily by reducing the three preceeding Forms, into a Progression of three Terms. 1. This Equation $xx=bx-bb$ may be reduced to this progression $\div x. a. b. x$; for $xx-ax=bb$; § adding ax to each Part, it is $xx=bx-bb$. 2. This Equation $xx=bx-bb$, is reducible to this progression $\div x. b. a-x$. for $ax-xx=bb$, † adding xx to each side, it is $ax=bb+xx$; taking away bb from each Side, it is $ax-bb=xx$. 3. This Equation $xx=bx-bb$, is reducible to this progression $\div x+a. b. x$; for $xx+ax=bb$: ‡ taking away ax from each side, it is $xx=bb-ax$. The mean Term b is known in each of these three Progressions, and in the first and third, the Difference of the Extremes is known, which in the first is $-a$, and in the third $+a$. In the second, the Sum of the Extremes is known. With these all the other terms of the progressions may be found, as will appear by the three following Problems.

- 44 PROBLEM 1. LET there be a Progression of three Lines $\div x. b. x$. the mean Term is b , and a is equal to the Sum of the Extremes x and x . required to find the Extremes.

Take AB equal to a , the Sum of the Extremes. From C, the Middle of AB as a Center, and with the Extent of AC or CB, describe a Circle. Upon B raise perpendicularly BD equal to the mean b ; and thro' the Top D draw DG Parallel to AB, and from E where DG cuts the Circle, let fall a Perpendicular FE, upon AB, which is equal to BD, hence $EF=b$; then seeing that $\div AF. EF. FB$. the Extremes are AF and FB; therefore

* B. 4. n. 29. § B. 3. n. 58. † B. 3. n. 57.
‡ B. 3. n. 57.

fore if x be the greater Extreme, FB which is the lesser Extreme, is equal to x .

PROB. 2. LET there be this Progression of three Lines $\div\div x+d$. b . x . or $\div\div x-d$. b . x . the mean b is known, and the Difference of the Extremes $x+d$ and x . To find the Value of x . 45

DRAW AB equal to d ; and upon B raise Perpendicularly BD equal to b ; then from C the half of AB, and with the Extent of CD, describe a Circle. Prolong AB at each end even to the Circumference of the Circle, after which AE, or BF= x . For $\div\div x+d$. b . x . If the Progression were $\div\div x-d$. b . x . the same thing must be done. But in the Case where $x=EB$, the Lesser Term is BF equal to EB-AB, or to $x-d$. It is evident that the difference between $x-d$ and x , is d . When the Sign is +, the Quantity x , is EA, to which is added the difference d , which gives the greater Extreme. When the Sign is -, d is subtracted from EB or FA which is the Value of x .

PROB. 3. LET there be this Progression of three Lines $\div\div d+x$. b . d , the mean Term b , being known, to find the Difference of the Extremes $d+x$ and d , which is x . 46

UPON B, the Extremity of AB, equal to d , raise Perpendicularly BC, equal to b ; thro' A and C draw a right Line, upon which at the Point C draw a Perpendicular which is CD; therefore the Angle ACD is a right one. Produce AB even till it cut CD; the Line BD-AB is equal to x ; for having divided AD into two equal parts at the Point K, and with the Extent of AK or KD describ'd a Circle, it will pass thro' C; hence $\div\div AB$. (or d .) BC. (or b .) BD: Wherefore $BD=d+x$, which is the third Term: having then subtracted from BD the Line DE equal to AB, the Remainder is equal to x ; whose value was required.

CHAP. IX.

OF the geometrical Construction of a Locus or Place.

A geometrical Place is the Line by which a Problem is 47 constructed, and as there are various Sorts of Lines, so there are different sorts of Places. When only a right Line is required, it is called a Place for a right Line. If a Circle must be used, then it is a Place for a Circle, &c. Indeterminate Problems are also of Place. Let us begin by a Place which is a Line. The Line LL is one, if there can be drawn from all its points the parallel Lines LN, LN, meeting a right Line AN; and having taken upon the Line AN a Point A at pleasure, each Line LN hath one and the same Proportion to the Part AN, which they contain. For Example, if LL is a right Line, and it

it meets the right Line AN in A, it is evident that every right Line LN hath the same Proportion to each part of AN, which may

be expressed by this Equation $y = \frac{bx}{a}$. If the Proportion be supposed to be as that of a to b , and that the Indeterminate Lines are $LL=x$, and $NN=y$. For since that $a::b::x, y$; then the Product of b by x , which is bx divided by a , is the Value of y ; hence $y = \frac{bx}{a}$. This Equation shews that the Place

is a Line; for 'tis only a Line that can have this Property, viz. That all Lines drawn from AL upon AN, are to AN as a is to b . Hence it appears that the Problem which required to find LL is a Place; and the Problem admitting of different Solutions, is therefore indeterminate, and the Place can be only a right Line; for there is only that which place LL possesses, which hath the Properties of the right Line.

49. A Problem is a place of a Circle, and indeterminate, when it is proposed to find a Line having its Square equal to a Plane; for then known Quantities must necessarily be supposed, as for Example, that the Sides of the Plane are a and b ; that $AB=a$, and $BD=b$. $AD=a+b$, and $AC=CD$. Upon C as a Center describe a Circle, and upon B draw the Perpendicular BE,

then $AB \cdot BE \cdot BD$; then $ab = BE^2$; hence if $BE=x$, then $ab=xx$, or $\frac{ab}{x} = x$. The Problem is indeterminate: For

whatever Ratio I suppose to be between the parts of AD, the Square of the Line which will fall Perpendicularly between the two Points A and D, will always have its square equal to the Plane of the parts of AD, that is, that $ab=xx$, or $\frac{ab}{x} = x$.

The Problem may therefore have innumerable Solutions. You see that to find an Equation we were obliged to suppose the known Lines AB and BD, otherwise it could not be done. There is only the Circle which hath always this Equation in all its parts.

C H A P. X.

50. SOLID Problems are not solvable by the Knowledge of these Elements, for there are only simple Equations, or those of two Degrees, which can be expressed Geometrically, with the Rule and Compass.

Ir

It has been shewn how to solve Equations of one or two Degrees by the Rule and Compasses, *that is*, by drawing right Lines and describing Circles, but this will not do, when they have more Degrees. See here an Example, in the trisection of an Angle which is a noted Problem. An Arc of a Circle being given, it is proposed to divide it into three equal parts, and consequently to find a third Part of the proposed Angle, whose measure is the third Part of that Arc. The Chord BE of the Arc BCDE is known, it is proposed to divide this Arc into three equal parts, suppose the thing as done, that $BC = CD = DE$; draw CF parallel to DH; hence $CF = DH$: and since that $CG = DH$, therefore $CF = CG$; hence the Triangle FCG is Ifoceles. The Angles BCG and HGC are equal; the Measure of BCG is half the Arc BK, and that of HGC is the half of KE, more the half of BC or DE equal to BC. * Now BK is equal to KD; then these two Angles having the same Measure are equal: Therefore the Triangle CBG is Ifoceles. It hath one Angle common with FCG; these two Ifoceles are therefore Similar. BAC is also Ifoceles, and has an Angle common with CBG, *viz.* BCG; these three Triangles are therefore Similar. † These three Triangles BAC, CBG, FCG being Similar, then $\frac{AB}{BC} = \frac{BC}{CG} = \frac{CG}{GF}$, ‡ then if $AB = 1$, and $BC = x$, according to what was said above, § $\frac{1}{x} = \frac{x}{xxx}$; then $FG = xxx$. The known Line EB I call b . Now $CD = FH = BC = BG = HE$; then $BG + FG + GH + HE$ is triple of BC; therefore there is wanting the Value of FG to make BE or b the triple of BC. Now $FG = xxx$, therefore $b + xxx = 3x$, or $xxx = 3x - b$. This is as far as we are able to proceed in this Problem. But we cannot find the unknown Quantity by help only of these Elements, it being only known that its Cube xxx is equal to three times itself, *that is*, to $3x$, the known Quantity b being deducted. This Problem is easily solved Mechanically. Let the Arc BC, be the Measure of the Angle BAC, which is to be divided into three equal Parts. Produce the Diameter CF towards E, apply a Ruler upon B, and upon the produced Part of CF, find in the Circle such a Point D, that the Line AD may be equal to DE; which done, I say that the Arc DF is the third Part of BC, or that DAF is the Third of BAC; which I demonstrate thus, ADE and DAB are Ifoceles; then $DBA = BDA$, and $DAE = DEA$: the exterior Angle BDA, is equal to the two Interior ones DEA and DAE; therefore DBA is equal to them both, and consequently is double either of them. The exterior Angle BAC is also equal to the two Interior ones DEA (or its equal DAF) and to DBA, therefore it is triple of DAF, the half of DBA; Q. E. D.

D d

Several

* B. 2. n. 39 & 52.

† B. 2. n. 83 & B. 4. n. 8.

‡ B. 4. n. 10.

§ Sup. n. 21.

Several other things concerning Equations might be added; but here is sufficient, for to make use of what could be said more, requires a Knowledge in the Doctrine of Curve Lines.

C H A P. XI.

- 52 **T**HE Solution of a Problem may be attempted two different ways. The First is only an Application of the former Part of these Elements, which discovers some course Particular to the Problem in hand, and which cannot be used in any other. The Second, is the Order prescribed by the Method which we have just taught, according to which the Thing sought is found in a much more excellent Method, as it extends universally to every Problem. Here follows an Example of each.

PROBLEM 1.

- 53 **BAC** is *Ifoceles*. It is required to divide the Sides *AB*, *AC*, by a Line Parallel to the Base, so that the Parallel may be equal to the Remainder of the Sides, that is, supposing the Thing done, that $DB = DE$.

METHOD 1.

SUPPOSE the Thing done, viz. That $BD = DE$, then the Triangle *BDE* is *Ifoceles*; hence the Angles *DBE*, and *DEB* are equal. Now the Angles *CBE* and *EBD* are also equal. Wherefore *EBC* and *EBD* are equal; therefore the Line *BE* divides the Angle *DBC* equally in two. From whence I know that in an *Ifoceles* Triangle, such as *BAC*, if the Angle *ABC* be divided into two equal Parts by a right Line *BE*, and there be drawn thro' *E* a Parallel to *BC*, it will be equal to *DB*; therefore by this property of the *Ifoceles* Triangle, I find the Way to solve the proposed Problem; but this Method is, you see, particular and proper to this Problem only.

METHOD 2.

- 54 **SUPPOSING** the Thing done, the known Line *AB* I call *a*, and *d* the Base *BC* which is known also, and the unknown Quantity *AE* which is sought I call *x*; hence as $EC = a - x$, also $DE = a - x$, it is evident that $a : d :: x : a - x$, then $aa - ax = dx$. Add to each Side ax , and it becomes $aa = dx + ax$; suppose $c = d + a$, hence $cx = dx + ax$; and consequently putting *cx* instead of $dx + ax$, I have $aa = cx$; then $\div$

c. a.

c. n. 21. Therefore the Business is only to find a third Proportional to the two known Lines *c* and *a*, which has been taught. *B. 4. n. 23.* This second Analytical Method is universal, and not particular to this Problem only.

PROB. 2. THE Sum of any two Sides of a Triangle *ABC* being known, to find each Side thereof. 55

LET $AB = x$, and $AB + BC = a$, and $AB + AC = b$, and $BC + AC = c$; then $BC = a - x$, and $AC = b - x$. Now $AC + BC = c$; then $a - x + b - x$, or $a + b - 2x = c$. Then adding $2x$ to each Side, it will be $a + b = c + 2x$. Taking away c from each, it is $a + b - c = 2x$. Therefore to find the Value of x , join the two Lines a and b together, from their Sum subtract c ; then is half the Remainder the Value of x .

PROB. 3. ONE of the Sides of a right angled Triangle is a , and the Other is x , whose difference with the Hypotenuse is b . To find the Value of b . 56

THE Hypotenuse is $x + b$; then $aa + xx = xx + 2bx + bb$. Taking away xx from each Side; $aa = 2bx + bb$; also subtracting bb , it will be $aa - bb = 2bx$; taking $cc = aa - bb$, * it will be $cc = 2bx$. Then $\div 2b$. *c. x*. The Business is therefore only to find a third Proportional to the two known Quantities, $2b$ and c .

PROB. 4. THE Hypotenuse of right angled Triangle is a , and the Difference of the two Sides b . To find the Sides. 57

LET the lesser Side be called x ; then the Greater is $x + b$; hence $aa = 2xx + 2xb + bb$. Subtract bb equally from each Side and $aa - bb = 2xx + 2xb$, or half aa - half $bb = xx + xb$. In the Place of half aa - half bb , I put the Square cc , which I find to be equal thereto. Then is $cc = xx + bx$, or $cc - bx = xx$; hence $\div x + b$. *c. x*. Hence may be found the Value of x .

PROB. 5. THE Hypotenuse of a right angled Triangle is a , and the Sum of the two Sides b . To find the Sides. 58

LET one of the Sides be called x ; then the other is $a - x$: Hence $aa = bb - 2bx + 2xx$. I add $2bx + aa = bb + 2xx$. I also subtract bb , and I have $2bx + aa - bb = 2xx$. Put dd in the Place of its equal $aa - bb$, then $2bx + dd = 2xx$. Divide the whole by 2, in order to which find cc , equal to the Half of dd ; + hence $bx + cc = xx$: Then $\div x$. *c. x*. Then the Value of x may be easily found.

PROB. 6. THE Area or Superficies of a right angled Parallelogram is aa , its lesser Side x , and $x + b$, the greater Side, required to find the two Sides. 59

THE Product of x by $x + b$ is the Area of the Parallelogram; then $xx + xb = aa$: hence $\div x$. *a. x + b*. This Problem is consequently Solved like the preceeding one.

PROB. 7. THE longest Side of a right angled Triangle is x , its difference with the Hypotenuse is a , which is therefore $x + a$, 60

D d 2

the

* *Sup. n. 24.*

+ *Sup. n. 26.*

the Difference of the Hypotenuse with the other Side which is lesser than that, is b, hence the Side is $x+a-b$. Required to find the three Sides.

THE Squares of the two Sides x and $x+a-b$ are equal to that of the Hypotenuse, which is $x+a$, hence $2xx+2ax-2bx-2ba+aa+bb=xx+2ax+aa$. Add $2ba$ to each Side, and it is $2xx+2ax-2bx+aa+bb=xx+2ax+aa+2ba$. Subtract $xx+2ax+aa$ from each part, and it is $xx-2bx+bb=2ba$, subtract $+bb$, and it is $xx-2bx=2ba-bb$: Taking d equal to $2b$, and cc equal to $2ba-bb$, it will be $xx-dx=cc$; hence $\div x. c. x-d$. Which is easily Solved. *

- 61 PROB. 8. *The Perpendicular drawn from the right Angle of a right angled Triangle upon the Hypotenuse is b; the Difference between the two Parts or Segments of the Hypotenuse is a. To find the Segments.*

LET the lesser Segment be x ; then the Greater is $x+a$. Now $\div x+a. b. x. \dagger$ Then the Problem is solved as has been taught. $\dagger$

- 62 PROB. 9. *THE Line AB is divided in one of its Points, as C; it is required to produce it to D, so that the Rectangles comprehended under AD and BD may be equal to the Square of CD.*

SUPPOSE the Thing done. Let $AC=a$, and $CB=b$, and $BD=x$. It is required to find the Value of BD , or x . Multiply AD , or $a+b+x$ by BD , that is, by x , which makes $ax+bx+xx$, which Product according to the Question is equal to the Product of CD , or of $b+x$ multiplied by itself; that is, $ax+bx+xx=bb+2bx+xx$. From both Members of this Equation, subtract $bx+xx$ and there remains $ax=bb+bx$; transpose bx to the other Side, in order that the unknown Quantity bb may remain alone, and then $ax-bx=bb$. To reduce this Equation to sim-

pler Terms, divide it by $a-b$, then $x=\frac{bb}{a-b}$, which Equation is

resolved into this Progression $\div a-b. b. x$, the two first Terms of which being known, the Third x , which is sought, is also known; therefore to construct the Problem produce the Line AB equal to x .

THE Solution of each Problem affords a new Theorem: For according to what has been proved, the Square of BD , the produced Part of a Line, more CB part of that Line, is equal to the Rectangle made of AD and BD ; and the produced Part is the third Term of a Progression, whose first Term is $AC-BC$, and the Second BC . The most part of Theorems are deduced from Algebra, which is a fruitful Source of truth.

- 63 PROB. 10. *THE Line AB is divided in C. It is proposed to divide it again in D; so that the Rectangle of $AC+DC$ by CD , may be equal to the Square of DB .*

Suppose

* Sup. n. 38 & 41. $\dagger$ B: 4. n. 28.

$\dagger$ Sup. n. 38. & 43.

SUPPOSE the Thing done. Then to find the Value of CD ; let $AC=a$, and $CB=b$, and $CD=x$; hence $DB=b-x$. The Rectangle of AD by CD is $ax+xx$, and the Square of DB is $bb-2bx+xx$; then according to the Question $ax+xx=bb-2bx+xx$. Subtract xx from both Sides, and $ax=bb-2bx$. Transpose $2bx$ to the other Side, in order that the known Quantity may be alone, $ax+2bx=bb$; divide this Equation by $a+2b$,

then is $x = \frac{bb}{a+2b}$; turn this Equation into a Proportion $\frac{bb}{a+2b} ::$

$a+2b$. b . x . the two first Terms are known, therefore x is known also; hence by taking upon CB the Line CD equal to x , there will be done what was required.

PROB. 11. THE right Line AB is divided at C , the infinite Line BD is Perpendicular upon AB ; required to draw from A , a Line AD upon BD , so that $AD=BC+BD$. 64

SUPPOSE the Thing done, and that $AB=a$, and $BC=b$, and $BD=x$, the Line required. According to the Question $AD=BC+BD$; wherefore $AD=b+x$. Now since that ABD is a right Angle, the Square of AD or of $b+x$, which is $bb+2bx+xx$, is equal to those of AB , and of BD , which are aa and xx . Hence $bb+2bx+xx=aa+xx$; taking away xx from both Sides, it is $bb+2bx=aa$. In order that the unknown Quantities may be all on one side, transpose bx , and it will be $2bx=aa-bb$, which

divided by $2b$, gives $x = \frac{aa-bb}{2b}$, which Equation is there-

fore reducible into this Proportion, $2b$. $a+b :: a-b$. x , whose three first Terms being known, the Fourth x ; which was sought, is also known.

PROB. 12. Two parallel Lines AB and CD , are given by Position with CF , as also the Points F and E . It is required to draw FD intersecting AB and CD , and produced if needful, so that AB be to ED , as AF is to AG . 65

LET $AF=a$, $CF=b$, $CE=c$, $AG=d$, and $AB=x$. I suppose the Thing done. Therefore according to the Hypothesis a . $d :: x$. ED . Then x multiplied by d , and the Product which

is dx divided by a , the Quotient $\frac{dx}{a} = ED$. The two Triangles AFB and FCD are Similar; a . $x :: b$. CD . Now $CD=$

$CE+ED$, and therefore $CD=c+\frac{dx}{a}$. The Product of the

Extremes a and $c+\frac{dx}{a}$, which is $ac+dx$, is equal to that of the Means: Then $ac+dx=bx$. Transpose $+dx$, and it is $ac=$

$=$

$=bx-dx$. This Equation divided by $b-d$, gives $x = \frac{ac}{b-d}$,

which Equation is reducible to this Proportion $b-d :: a :: x$, whose three first Terms are known.

66 PROB. 13. To measure an inaccessible Height AB , and its distance AC , by the Help of two Sticks CD and EF .

SUPPOSE $AC=x$, $AB=y$, $CE=b$, $GD=a$, and $FH=c$; then seeing that the Triangles BAF and GDF are Similar, $AB :: GD :: FA :: FG$; but $FA :: FG :: FI :: FK$, upon account of the Parallels GD , and BA ; then $AB :: GD :: FI :: FK$, or their equals AE and CE , hence $y :: x+b :: b$, consequently $by=ax+ab$. The two Triangles BHF and BGD are Similar; then as BD is to BF , or IK to IF , or AC , to AE , so likewise GD is to HF ; hence $AC :: AE :: GD :: FH$, or $x :: x+b :: a :: c$, therefore $xc=ax+ab$; taking ax from both Sides, there remains $xc-ax=ab$;

dividing each Member by $c-a$, it will be $x = \frac{ab}{c-a}$; hence $c-a$

$a :: b :: x$. The three first Terms of this Proportion are known; therefore the Fourth which is required is also known. Seeing it appeared above that $by=ax+ab$; and $xc=ax+ab$, it follows that $by=xc$, it being equal to one and the same Quantity $ax+ab$, wherefore $b :: c :: x :: y$. Now x is known already: Therefore the three first Terms being known, the fourth Term y which is required, is known.

67 PROB. 14. Two Merchants put into Company 1200 Pounds, and they gained 3400 Pounds; the First took 700 Pounds, for both his Gain and Sum put in for two Months; the Second took 3900 Pounds, for his Gain and Sum put in for five Months. It is demanded what each Gained, and also the particular Sum put in by each?

THE Sum put in by both is $1200=a$. Let the Sum put in by the First be called x ; hence that of the Second is $a-x$. The Stock and Gain of the First is $700=b$: Then $b-x$ is the Gain of the First. The Stock and Gain of the Second is $3900=c$; therefore $c-a+x$ is the Gain of the Second. Now as the Stock of the First multiplied by its time, is to its Gain; so the Stock of the Second multiplied by its time, is to its Gain; that is, $2x :: b-x :: 5a-5x :: c-a+x$. The Product of the Extremes is equal to that of the Means; then $2xc-2ax+2xx=5ab-5ax-5bx+5xx$; and $2ax$ being added, and $2xx$ at the same Time subtracted from both Sides of the Equation, it will be $2xc=5ab-3ax-5bx+3xx$; add $3ax+5bx$ to both Sides, and it gives $2cx+3ax+5bx=5ab+3xx$. Suppose $d=2c+3a+5b$; hence $dx=5ab+3xx$; also take $f=5ab$, which subtract from each Side, and then $dx-f=3xx$. Then to reduce this Equation to a form which gives the Solution of the Question, suppose $d=3g$, and $f=3h$; hence $gx-h=xx$, find a Square equal

equal to b , which I call ll , wherefore $gx - ll = xx$; which Equation is reducible to this Progreſſion $\div g - x$. I. x , for $gx - xx = ll$, and conſequently $gx = ll + xx$, and $gx - ll = xx$, the Extremes of this Progreſſion are $g - x$ and x , whoſe Sum is g . Their difference will be found if upon $AB = g$ be deſcribed a Circle AGB , and there be raiſed a Perpendicular $BD = l$, and DG be drawn Parallel to AB , and from the Point E , be drawn the Perpendicular EF , it will cut AB at the Points required. The unknown Quantity which is ſought is x . Now as we in this Problem ſeek the Value of x in numbers, there muſt from the Square of CE (or CB , the half of AB , the Sum of the known Extremes,) be ſubtracted the Square $FE = ll$, which is alſo known, and there will remain the Square of CF , the Half of the Difference of the Extremes, whoſe Root added to CB will give the greater Term, and being ſubtracted it gives the Leſſer $x = 300$, the Stock of the Firſt; after which all the Reſt is eaſy.

ADVERTISEMENT.

THE two following Problems ſhew that an extenſive uſe may be made of this Method of ſolving Queſtions, which tho' ſeemingly as not appertaining to Geometry, yet their Solution depends upon our Knowledge herein.

PROB. 15. LET $ABCD$ be a Billiard Table. Required to ſtrike the Ball H after two Rebounds, with the Ball K . 68

THE Angles of Incidence and Reflexion are equal. According to this Principle, it is ſuppoſed that if K is pushed to the Point I it ought to reflect to L , and from L to H . The Point I is then required. Having drawn thro' K the Line EF , parallel to the Side CD ; and thro' the other Point H , the Line GH Parallel to BD . Let $KF = a$, $FB = b$, $BG = d$, $GH = f$, and $FI = x$: Then $BI = b - x$. The Triangles KFI and IBL being

Similar, $x :: a :: b - x$. $BL = \frac{ab - ax}{x}$, and by con-

tracting this Expreſſion $BL = \frac{ab}{x} - a$. Now $GB - BL =$

GL : Then taking away $\frac{ab}{x} - a$ from d , it will be $GL = d + d$

$\frac{ab}{x}$. The Triangles KFI and LGH are ſimilar: Therefore

$a :: x :: d + a - \frac{ab}{x}$, f : Therefore $af = dx + ax - ab$: Then adding

ab to each Side and dividing by $d + a$, it is $\frac{af + ab}{d + a} = x$; and

reſolving

resolving this Equation into a Proportion, it will be $d+a : f+b :: a : x$; that is, FI is a fourth Proportional to these three Lines BG + KF, GH + FB, and KF; which are known Lines: Wherefore FI is the Line sought, therefore the Point is I. Therefore if the Ball K is to touch the Ball H after two Rebounds, K must be struck so that it go to I, from whence it will rebound to L and from L to H; which is the Thing proposed.

69 PROB. 16. Two Points or two Places being given, with a Plane at a Distance from those two Points, required to find upon this Plane, a Point which must be passed thro' in order to go the shortest Way from one Place to the other. Or a luminous Object being given with a Mirrour, and the Eye placed out of the Mirrour. To find the Point of Reflection.

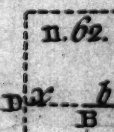
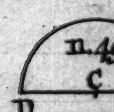
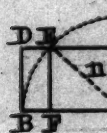
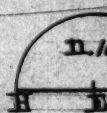
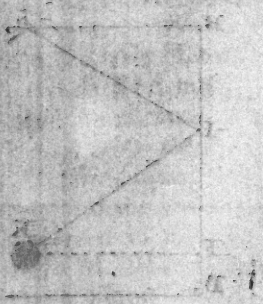
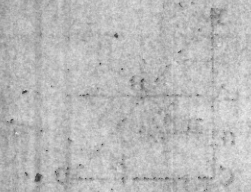
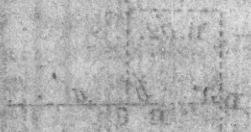
LET the given Points be A and E; and let also the Plane be BF. Required to find upon that Plane the Point D, such that the Line AD more DE, may be shorter than any other of the Lines drawn from that Point to the Plane, for Example, shorter than AG + GE, &c.

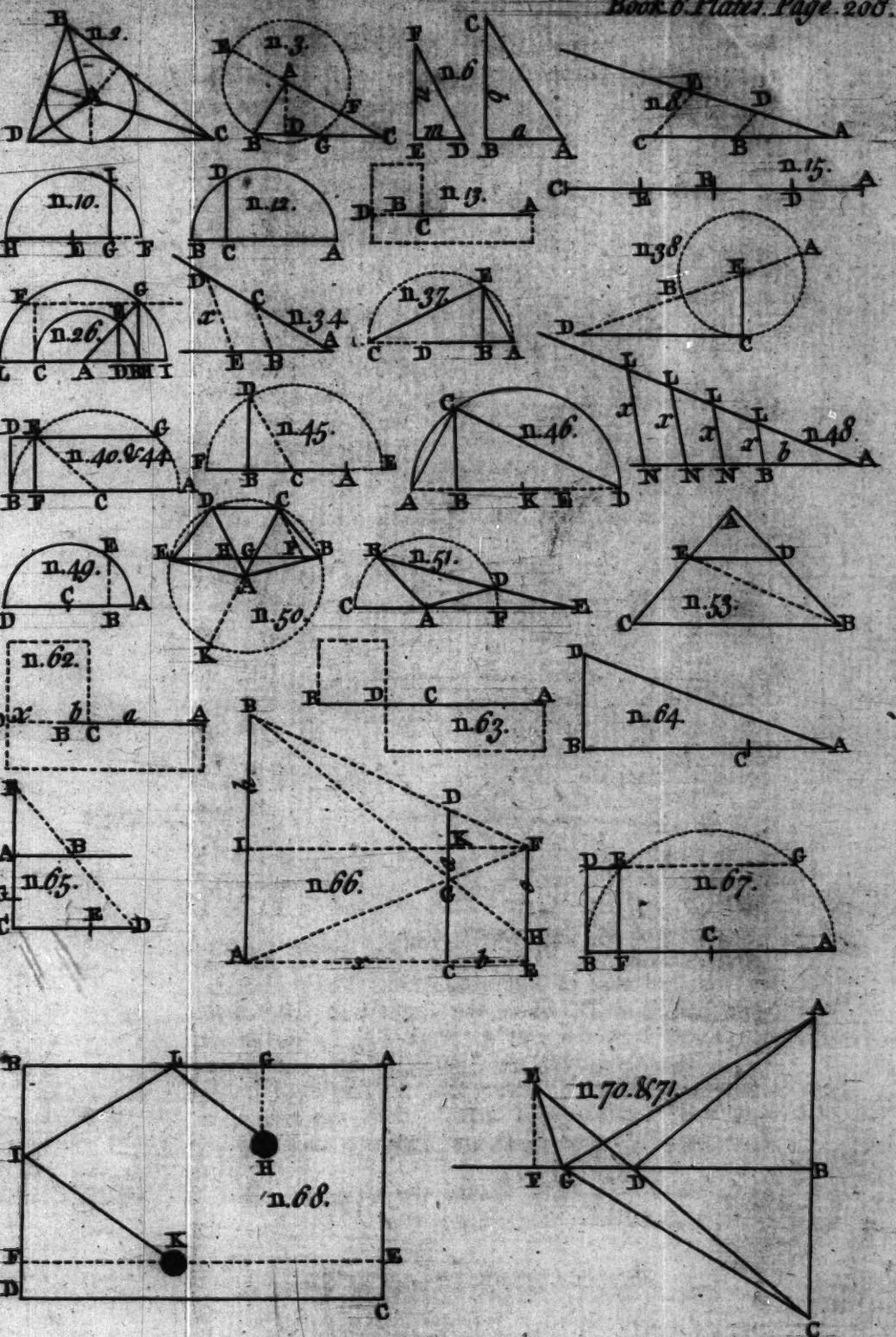
70 FIRST. Let there be drawn the Perpendicular AB produced even to C, so that AB = BC. And from the Point E, let there be drawn the Line CE; I say that the Point Sought is D: For the two Triangles ABD and DCB are equal, as are ABG and CBG: Then AD + DE = CD + DE, and AG + GE = CG + GE. But CD + DE is a right Line: It is therefore shorter than CG + GE. If the Point G were between B and D, the Demonstration would be the same. Observe that the two Triangles ABD and CBD being equal, they are Similar: Therefore the Angles ADB and BDC are equal; but EDF is equal to BDC: Then EDF is equal to ADB; hence as the Angle of reflection is equal to that of Incidence, if the Object is at E, and the Mirrour be BF, the Point of Reflection is D, and the Light will be reflected at A. It is there-

71 fore conveyed by the shortest Way.
SECONDLY. The Angle of Incidence EDF is supposed equal to that of Reflexion ADB. The Business is to find again the Point D, where the Light will fall reflected from the Object E to the Eye A, proceeding by the shortest Way. The known Quantities are AB = a, EF = b, and BF = d. The unknown is BD = x. Seeing that the Angles of Incidence and Reflection are supposed equal, then the two right angled Triangles ABD and EFD are Similar; then $\frac{a+b}{ad} : a :: d : x$.

Therefore $\frac{a+b}{ad} = \frac{d}{x}$: Hence the Value of BD is known.

THE End of the Elements of Geometry.





A N

INTRODUCTION TO CONIC SECTIONS.

CHAPTER I.

OF the curve Lines which represent the different Sections of a Cone. Their Names, and the easiest Method for finding their principal Properties.

In these geometrical Elements I have treated only of the right Line and the Circle; but as there are other Lines called curve Lines, which are now-a-days chiefly the Object of the most sublime Geometry, upon account of the wonderful and singular Properties which they discover, it will be proper to shew their *Genesis*, and some of their principal Properties. I shall treat only of the First and most known, which are those observed in cutting the Cone different ways, from thence, called *Conic Sections*. The Cone and its several Sections are shewn in P. B. Plate 8.

LET there be conceived a right angled Cone. 1. If it be cut by a Plane passing thro' its Axis, it is evident that the Section will be a Triangle, whose Sides are those of the Cone, and its Base the Diameter of the Circle which is the Base of the Cone. *In what follows I suppose the Plane which divides the Cone, to be Perpendicular upon the Plane of the Triangle.* 2. If the Section be Parallel to the Base of the Cone, it is evident the Section is a Circle. 3. If it be not Parallel to the Base, but intersects the Cone's Axis, and both its Sides, (*I mean the two Sides of the Triangle, which is the Section of the Cone by a Plane passing thro' its Axis*) this Section, or the Circuit of this Section will represent a curve Line called an *Ellipsis* or *Oval*. Such is the Figure x, formed in the Cone A by cutting it in the Direction of the Line M. See this Section PB. Plate 8. fig. 30. 4. If the cutting Plane divides only one of the Sides of the said Triangle, and that the Section be Parallel to the other Side,

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this

this Section or the Circuit thereof is a different Curve, called a *Parabola*. Such is the curve Figure Y, formed in the Cone B, by the Plane which cuts it, according to the Line N. See P. B. P. 8. fig. 32. If the Section be such that if prolonged it cross the other Side of the said Cone or Triangle being prolonged above its Vertex, this Section or the Circuit thereof is called an *Hyperbola*, such is the Figure Z, formed in the Cone C, by a Plane dividing it according to the Line O. See also P. B. P. 8. Fig. 33.

THERE are certain Terms which we necessarily use in treating of Conic Sections, which I thus explain. Let the Section of a Cone thro' its Axis be the Triangle BAC. Let $AP=x$, $PM=y$, $AD=a$, $BD=b$. Then $x:y::a:b$. Then $ay=bx$, which is the Equation, that expresses the Nature of the Triangle. Let AMB be a Semi-circle whose Center is N, and $AN+NB$, or AB is the Diameter. Let $AP=x$, $PM=y$, $AE=a$; hence $PB=a-x$. Wherefore $x:y::a-x$; therefore $ax-xx=yy$, is the Equation which expresses the Nature of the Circle. A Perpendicular such as MP drawn from any Point whatsoever as M of the Circumference upon the Diameter AB, is called an *Ordinate*, and the Part of the Diameter taken between its Extremity and the meeting of the *Ordinate*, is called the *Abscissa*. Let DEE be a given right Line, with F a Point out of that Line, I call this Line DEE the *Directrix*. Let AMN be such a Curve, that every Line Perpendicular upon DE falling upon M, one of the Points of the Curve, may have always one and the same Proportion with a second Line drawn from the Point M to the given Point F. This Curve is regular and may be described that is, all the Points thro' which it ought to pass may be found. To these Points, and these Lines, there are names given.

THE Point A is the *Vertex* of the Curve. The Point F is called the *Focus*. The Line AF produced is the *Axis*. The Lines which cross the Axis Perpendicularly, and which are comprehended between the Curve and its Axis, are called *Ordinates*. PM is an *Ordinate*. The Part of the Axis taken from the Vertex A, to the Meeting of an *Ordinate*, is called *Abscissa*. The peculiar Ratio which is between the Line AD, to AF, is that which makes the essential Difference between these three Conic-Sections. The Parabola, Ellipsis and Hyperbola. In these three Sections I shall call the Proportion that of p . to q . In the Parabola, 'tis always a Ratio of Equality; in the Ellipsis p is always greater than q ; and in the Hyperbola p is always less than q .

C H A P. II.

OF the Parabola, or Curve-Line which represents the Section of a right Cone, by a Plane parallel to one of its Sides.

DEFINITION I.

THE right Line DE is given, with the Point F out of that Line. DE is the directrix of the Parabola. The Line FD is Perpendicular upon the Directrix DE . If this Perpendicular be divided at the Point A , so that $FA=AD$, and that having drawn from each Point of the Curve as M , a Perpendicular upon DE , and another Line at the Point F , if these two Lines are equal. $MF=ME$, this Curve is called a Parabola.

It will be demonstrated that this Curve is the Conic-Section called a Parabola. I shall express the Proportion of Equality of EM to MF , by the two Letters p and q .

PROB. 1. To find each Point of the Parabolic Curve.

DE is the directrix Line which is given, and F the Focus. Divide DF into two equal Parts, hence A is the Vertex of the Curve. To find the other Points. 1. Thro' A draw a Parallel to DE , From which take $AX=FA$; and draw thro' D and X an indefinite Line. 2. Having afterwards drawn a sufficient number of Lines parallel to DE , which cut the Axis DF , it being prolonged, the Points of these Parallels thro' which the Curve passes will be easily found; the Point M , for Example, in the Line PO . With the Extent of PO and from the Focus F as a Center, describe a Circle which will cut PO in M ; then $AD:AX::p:q$, and $AD:AX::DP:PO$. Now $DP=ME$, and by construction $PO=FM$: Then $* ME:FM::p:q$. Hence M is one of the Points of the Parabola, according to the preceding Definition. Observe that the indefinite Line DX makes with DF an Angle of 45 Degrees; for DAX is a right Angle, and $AD=AX$: Therefore the Triangle is Isosceles; hence the Angle ADX is equal to AXD ; consequently each is 45 Degrees, or the Half of a right Angle.

DEF. 2. A Line always double FD , or quadruple FA , or AD , is called the Parameter of the Parabola.

LET the Parameter be called a ; the Ordinate PM be called y , and the Abscissa PA be called x . There is no occasion to be at the Trouble of explaining the Parameter any farther, than what is done by the Definition, which calls it thus, a Line quadruple of FA or of AD .

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* B. 3. n. 53.

THEOR.

THEOR. 1. THE Rectangle made of the Parameter and Abscissa, is equal to the Square of the Ordinate. Or that Ordinate is a mean Proportional between the Parameter and the Abscissa.

LET FA or AD (fig. preced.) be called b . The Abscissa AP hath been called x . Then PD or ME $= b + x$, and FP $= x - b$, or $b - x$, according as the Point P is found above or below the Focus F. The Parameter a is Quadruple AD or AF, and therefore of b ; hence $a = 4b$. It must be demonstrated that $\frac{a}{x} = \frac{y}{y}$; or that $ax = yy$. According to the first Definition MF $=$ EM;

but EM $=$ PA + AD $= b + x$. Then MF $= b^2 + 2bx + xx$. And since that FP $= b - x$; Then FP $= b^2 - 2bx + xx$. Now FM - EP

$=$ PM $= yy$. * Then $b^2 + 2bx + xx - b^2 + 2bx - xx = yy$. But $+b^2 - b^2 = 0$, and $+xx - xx = 0$; then the Remainder is $2bx + 2bx$, or $4bx = yy$. Now $4b$ is equal to the Parameter a ; therefore $ax = yy$ or $a : y :: y : x$; which was to be proved.

COROLLARY. THE Squares of the Ordinates of a Parabola are one to another, as the Parts of the Axis taken between its Vertex and the Meeting of these same Ordinates.

THE Parameter a is always the same; but x which is the Abscissa is lesser or greater, according as the Ordinate is nearer or farther from the Vertex A. Now seeing that it is always $yy = ax$, in whatever place the Ordinate is taken; then yy or the Squares of the Ordinates are one to another as x , or the Abscissa's. †

THEOR. 2. THE Curve or conic Section which has been defined and described, is a Parabola.

SVT is a Cone cut by a plane parallel to its side TV, according to the line AF: it must be proved that the Section QAR, both is, and hath the Properties of the Parabola.

THE Lines PM and QF are perpendicular upon AF, which I call the Axis of this Curve. PM and QF are therefore Ordinates: Consequently to prove that this Parabola QAR is that Curve which has been treated of, requires no more than to prove

that the squares MP and QF and of all the other Ordinates, are one to another as the Parts AP, AF, of the Axis which they intersect; for this being so, then according to the preceeding Corollary, the Curve QAR is the line required. Let us suppose the right Cone SVT to be cut at the Point M by a plane Parallel to its Base. This Section PME is then a Circle. Let SQT be another Circle Parallel to DME. The Line MP is the Ordinate of the Parabola QAR, and of the Circle DME, it being perpendicular as well upon ED as upon AF, forasmuch as it is the common Section of the Plane of the Circle and of that of the Parabola, which by Construction, divides that of the Triangle at

* B. 4. n. 78.

† B. 3. n. 54.

at right Angles. It is the same of QF, which is also an Ordinate, to the Circle as well as to the Parabola. Now $\therefore DP \cdot MP \cdot PE$; and so likewise $\therefore SF \cdot FQ \cdot TF$: Then $DP \times PE = MP^2$, and $SF \times FT = FQ^2$: Then $DP \times PE \cdot SF \times FT :: MP^2 \cdot FQ^2$

but $PE = FT$: Therefore $PM \cdot FQ :: DP \cdot SF$. Now because of the Similar Triangles DPA and SFA, $AP \cdot AF :: DP \cdot SF$: then $PM \cdot FQ :: AP \cdot AF$: Therefore by the preceeding Corollary, QAR is of the same Nature as the Line treated of in that Chapter, which is also a Parabola.

THEOR. 3 To draw a Tangent thro' any Point required, of a given Parabola.

LET AMB be a Semi-Parabola; required to draw the Tangent MT, from the Point M. Let there be drawn the Line FM from the Focus F to the Point M, and from the Point M the Line MP parallel to the Axis, and bounded by the Directrix DP. If there be drawn FP, and it be divided into two equal Parts, the Line MT drawn thro' the Point of Division will be the Tangent required; which is to be proved. The Triangle EMP is Isosceles; therefore MT is perpendicular upon FP. When a Line touches only in one Point it is a Tangent, which is the Case here; for every other Point that can be assigned in the Line TS, is not in the Parabola, but out of it; as for Example, N taken in TS above or below M, is out of the Parabola. For let there be drawn the Lines NF, NP, and NE, Parallel to the Axis, NP will be always greater than NE. Now since that TN is by Construction perpendicular upon FP, the Lines NF and NP are equal. NF is then greater than NE. consequently N is not one of the Points of the Parabola: for if it were, FN would be equal to NE, according to the Definition of the Parabola. But FN being greater, the Point N, is not in the Curve.

COROLLARY. IT is evident that if the Curve AMB represent the Section of a parabolical Mirrour, and that CM be an Incident Ray parallel to the Axis AF, the Line MF will represent the reflected Ray; the Angle of Incidence CMS being equal to the Angle of Reflection FMT: For the Angle CMS is equal to the opposite Angle TMP. Now by Construction, TMP is equal to TMF; therefore TMF is equal to CMS. Wherefore all the Rays which fall upon the Concave Surface of the Mirrour parallel to the Axis, will reunite at the Point F. and this occasions it's being called the Focus, or burning Point. Hence the Concave parabolical Conoid, is the properest Form for a good burning Mirrour.

C H A P. III.

OF the Ellipsis, or Line which represents the Section of a Cone, by a Plane cutting its two Sides, and which is not parallel to that of its Base.

DEF. 1. LET there be given the right Line DE for the Directrix of the Curve that is to be defined, with the Point F; out of DE. FD is perpendicular upon DE. If this Perpendicular is divided at the Point A, according to the Ratio of p to q , I suppose p greater than q ; and that having drawn a Perpendicular from each Point of the Curve, as M, upon DE, and another Line at the Point F, if the Line EM be always to FM as p to q ; this Line ought to be called an Ellipsis, because it will be proved to be the same as the Section of the Cone which is called by that Name.

PROB. 1. THE Directrix DE being given with F, which is a Point out of it, to find all the Points of the Ellipsis.

1. FD must be divided in A; so that AD. FA :: p . q .
 2. Thro' A draw a Parallel to the Directrix DE. and take AX equal to AF. Hence AD. AX :: p . q . 3. Thro' D and X draw an indefinite Line; after which dividing the Axis by as many Parallels as you please, it will be easy to find in those Parallels, the Point thro' which the Curve passes. From F as a Center, and with the Extent of PO, describe a Circle which will cut PO in M. By reason of the similar Triangles DAX and DPO, the Line PD, or its equal EM is to PO or FM, as AD is to AX; therefore EM. FM :: AD. AX :: p . q . The Point M is therefore a Point of the elliptical Curve.

PROB. 2. To find the A of this Curve.

LET the Line DX be prolonged *ad inf.* and upon the produced Part of AD at the Point F let there be drawn a Line making an Angle of 45 Degrees, and continued even till it meets the indefinite Line DX. From the Point of Intersection x , let there be drawn the Perpendicular xa upon Fa. Then in the Triangle Fax, the Angle Fxa is also of 45 Degrees: therefore the Triangle Fax, is Isocetes. Now AD. AX :: Da. ax; and seeing that Fax is Isocetes, $ax = Fa$: Then AD. AX :: Da. Fa. But AD. AX :: p . q . Then Da. Fa :: p . q . Wherefore a is one of the Points of the Curve, as is evident.

DEF. 2. THE transverse or longest Axis of an Ellipsis is Aa. The Point C which divides Aa into two equal Parts, is the Center. The Line BG which cuts Aa at right Angles, is the conjugate or lesser Axis. Having taken sa equal to FA, the Points F and f are the Focii. 2. The Perpendiculars drawn from any Point whatsoever of the Curve to one of the Axes, are called Ordinates.

Hence

HENCE PM being an Ordinate to the Axis Aa , the Line AP is an Abscissa. 3. The third Proportional to the two Axes is called the Parameter of the first Term of the Proportion.

THEOR. 1. THE Distance Ff of the Focii F and f, is to the greater Axis Aa , as q is to p . (fig. preced.)

It has been demonstrated that $q. p::AX. AD::Fa. Da$: Then $Fa-AX. Da-AD::q. p$. Now since that $fa=FA=AX$: Then $Fa-AX=Ff$. But $Da-AD=Aa$: Therefore $Ff. Aa::q. p$: Q. E. D.

THEOR. 2. AC the Half of the greater Axis Aa , is a mean Proportional between FC, the Half of the Distance of the Focii, and the Line CD.

THE Half is to the Half, (fig. preced.) as the whole is to the Whole; wherefore $AC. FC::Aa. Ff$. But by the preceding Theorem, $Aa. Ff::AD. AX$. Then $AC. FC::AD. AX$; or $FC. AC::AX. AD$, and $FC. FC+AX::AC. AC+AD$. Now by construction $FA=AX$; hence $FC+AX=FC+FA=AC$, and $AC+AD=CD$. Putting then AC in the Room of $FC+AX$ and CD in the Place of $AC+CD$, it will be $FC. AC::AC. CD$. Wherefore $∴ FC. AC. CD$; Q. E. D.

PREPARATION.

FOR the Demonstrations of the following Theorems. (same Figure.)

LET $AC=d$, and $CF=g$: Then FA or $AX=d-g$, and Fa or $ax=d+g$. Now AC or d is a mean Proportional between CF or g and CD , by the preceding Theorem. Then $∴ g. d. CD$. Then d multiplied by d , and the Product dd divided by g , the Quotient $\frac{dd}{g}$ is equal to CD . Let $CP=x$, and $PM=y$:

Then $PF=x-g$, or $g-x$, and DP , or $ME=\frac{dd}{g}-x$, when the Point P is above the Center C. By the Generation of the Ellipsis. $d. g::\frac{dd}{g}-x. MF=d-\frac{gx}{d}$, * the fourth Term of this Proportion is MF . When the Point P, is below the Center C, then $PF=x+g$, and DP , or $ME=\frac{dd}{g}+x$: Then, as above,

$$MF=d+\frac{gx}{d}.$$

THEOR. 3. THE Square of any Ordinate whatsoever as PM at the greater Axis Aa , is to the Rectangle $AP \times Pa$, the Parts of that Axis, as the Rectangle $AF \times Fa$, Parts also of that Axis,

* B. 3. n. 60.

is to the Square of AC or Ca, the Half of that same Axis. (same Fig.)

AC=d, CF=g, PC=x: Then AP=d-x, and Pa=d+x. Hence AP×Pa=dd-xx. So likewise seeing that AF=d-g, and Fa=d+g: Then AF×Fa=dd-gg. Hence supposing PM=y, it is then required to demonstrate that yy.dd-xx::dd-gg.dd. Now as the Product of the Extremes is equal to that of the Means, that is, yydd=d²-ddxx-dd²gg+ggxx. The

Triangle FMP being right angled FM²-PF²=PM²=yy. Now FM=d - $\frac{gx}{d}$; hence FM²=dd-2gx + $\frac{ggxx}{dd}$, and PF=g-x:

Then PF²=gg-2gx+xx. Then PM, or yy=dd-2gx + $\frac{ggxx}{dd}$ - gg+2gx-xx. Now cancelling the Terms which destroy one

another by + and -, it will be yy=dd + $\frac{ggxx}{dd}$ - gg-xx; and all the Terms on both Sides multiplied by dd, it will be ddyy=d²+ggxx-ggdd-ddxx; then &c.

THEOR. 4. LET BC=b, the Rectangle AF×Fa, or dd-gg is equal to bb the Square of BC, half the lesser Axis BG. (same Figure.)

It must be proved that bb=dd-gg. BC=b is an Ordinate which is at the Center; hence x=0, and d=d, and y=b; hence yy=bb. Now by the preceeding Theorem yy.dd-xx::dd-gg.dd. Then substituting bb instead of yy, and cancelling -xx, it will be bb.dd::dd-gg.dd; or bb.dd-gg::dd.dd; hence bb=dd-gg; which was to be proved.

THEOR. 5. THE Square of MK an Ordinate of the lesser Axis BG is to the Rectangle BK×KG parts of that Axis, as the Square of the greatest Axis Aa is to the Square of the Lesser BG. (same Figure.)

THE greater Axis Aa is equal to AC+Ca, or to 2AC; therefore to 2d. The Half of the Lesser Axis BG is called b; hence 2b=BG. The Line CP has been already called x. Now the Ordinate MK is equal thereto, which is therefore x. The Line PM the Ordinate of the greater Axis hath been called y. Hence CK which is equal thereto, is also y; wherefore b-y=BK, and b+y=KG, and bb-yy=BK×KG. The Square of the greater Axis Aa or 2d, is 4dd. That of the lesser Axis BG or 2b, is 4bb. Therefore this is what is to be demonstrated. xx.bb-yy::4dd.4bb. According to the third Theorem, yy.dd-xx::dd-gg.dd. By the Fourth bb=dd-gg. Then Substituting bb instead of dd-gg, it becomes yy.dd-xx::bb.dd. Then yy.dd-xx::4bb.4dd; and the Extremes and Means multiplied, it will be 4ddy=4bbdd-4bbxx, or 4bbx=4bbdd-4ddy; and turning this

this Equation into a Proportion, it will at last be $xx. bb-yy:: 4dd. 4bb$.

THEOR. 6. THE Parameter is to the Diameter as the Square of its Ordinate; is to the Rectangle comprehended under the Parts of the Diameter taken from the Section of that Ordinate. (same Figure.)

AC the Half of the longest Diameter Aa has been called d ; and b that of the shortest Diameter BC. Hence $BG=zb$. Let the Parameter be called a ; then according to the Definition, $2d$.

$2b::zb. a$; then $\frac{4bb}{2d}$ or $\frac{2bb}{d}=a$, and both Members of this

Equation multiplied by d , it will be $2bb=ad$, and divided by 2, it is $bb=\text{half } ad$. By the third and fourth Theorems above, it appeared that $yy. dd-xx::bb. dd$. Putting half ad in the Room of bb , it will be $yy. dd-xx::\text{half } ad. dd$. Each Term of the Ratio of half ad to dd divided by d ; gives half a , and d ; and multiplying them by 2, it becomes $a, 2d$; therefore $yy. dd-xx::a. 2d$, or $a. 2d::yy. dd-xx$; which was to be proved.

THEOR. 7. THE Squares of the Ordinates are one to another, as the Rectangles of the Parts of the Axis made by the Meeting of the same Ordinates.

Y and m are two different Ordinates, and n a Part different from PC. By the preceeding Theorem, $a. 2d::yy. dd-xx::mm. dd-nn$. Therefore $yy. mm::dd-xx. dd-nn$; which was to be proved.

THEOR. 8. THE Conic Section or Curve which has been defined and described, is an Ellipsis.

LET the Cone mon be cut by a Plane making an oblique Angle with its Axis, and which intersects both its Sides. The Section DMAD, which is an Ellipsis, is the same Line as that whose Properties we treated of. Let there be conceived two parallel Circles EMF and GRH, whose Diameters intersect that of the Curve DMAD, in I and in q . If from the Points M and R, where those Circles cross that Curve, there be drawn right Lines to the Points I and q ; it is evident that these Lines are Ordinates common to both the Circles and to the Curve DMAD, forasmuch as Rq is Perpendicular as well upon HG as upon AD, being the common Section of the Plane of the Circle and of the Ellipsis, which consequently intersects that of the Triangle at right Angles. It is the same of MI. The Triangles AqH and AIF are Similar; then $Aq. FI::Aq. AI$. The Triangles DEI and DGq, are also Similar; then $Gq. EI::qD. ID$. Each Term of the first Proportion multiplied by its corresponding Term in the Second, the Products being compounded of equal Ratio's, are Proportional. $Hq. FI::Aq. AI, Gq. EI::Dq. DI$. Hence $Hq \times Gq. FI \times EI::Aq \times Dq. IA \times DI$, because of the Circles EMF and GRH, according to these Elements

$\frac{-2}{F} qR = Hq \times qG$, and $\frac{-2}{F} IM = FI \times IE$: Then $\frac{-2}{F} qR. IM; Aq \times Dq,$

D_g. AIXID. Then by the preceeding Theorem, this Ellipsis DAMD is the same as that Line whose Properties we have demonstrated, and consequently it is an Ellipsis.

PROB. 3. To describe an Ellipsis by a continued Motion.

LET there be given two Points H and I as Focii of an Ellipsis, and the Line LK as the transverse Axis or longest Diameter. To describe this Line, take a Thread whose length is equal to the longest Diameter of the Ellipsis, and tie it together by its two Ends at the two Points H and I, which will guide the Hand to a moveable Point which will leave a Trace in moving round about H and I. This Trace is an Ellipsis; which must be demonstrated. By the Construction of this Curve whatsoever it be, the two Lines HE and EI taken together, are equal to the longest Axis KL; hence there is no more to prove than that the Ellipsis which we have treated of, hath this Property. Let there be a Geometrical Ellipsis, whose longest Diameter is Aa, the Focii are F and f. It must be demonstrated that $FM + Mf = Aa$. Let the Diameter Aa be produced on each Side, so that AD. AF, or AX::p. q, and likewise at. af::p. q, and that therefore at=AD. Thro' t let there be drawn a Parallel to DE. The Proportions of FM to PD are the same as fM to Pt. By the Definition, $\frac{FM}{fM} = \frac{PD}{Pt} :: q. p :: AX. AD$. It has been proved that $Ff. Aa :: AX. AD$: * therefore $Ff + 2AX. Aa + 2AD :: AX. AD$. Now $Ff + 2AX = Aa$, and $Aa + 2AD = Dt$. Then $Aa. Dt :: AX. AD$. $FM + fM. PD + Pt :: AX. AD$. and $Aa. Dt :: AX. AD$. Then $FM + fM. PD + Pt :: Aa. Dt$. But $PD + Pt = Dt$; then $FM + fM = Aa$: Which was to be proved.

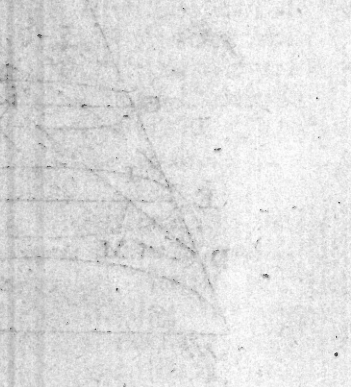
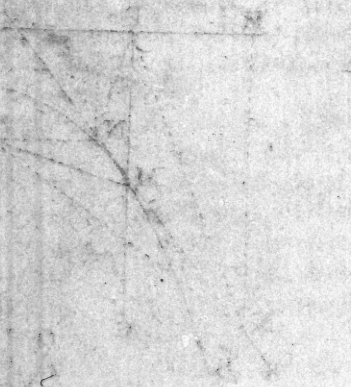
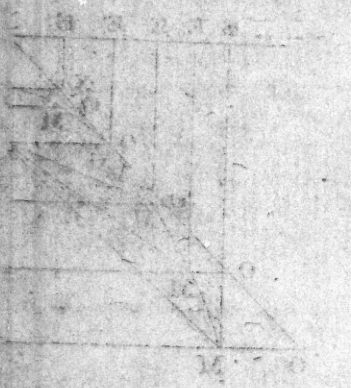
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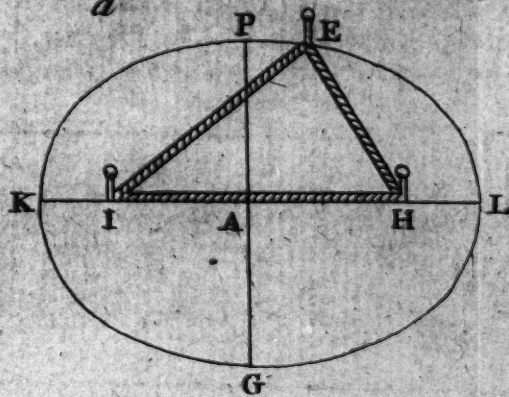
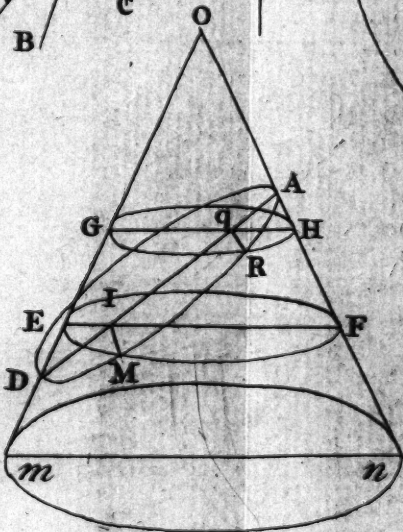
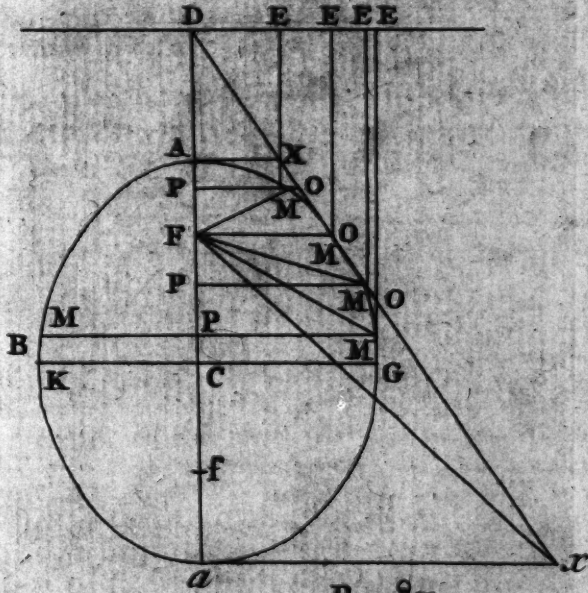
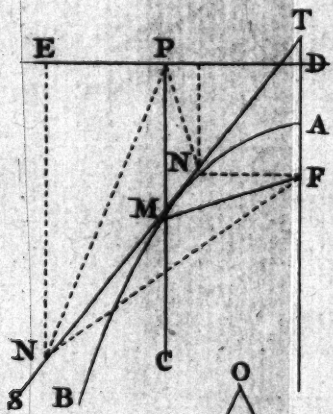
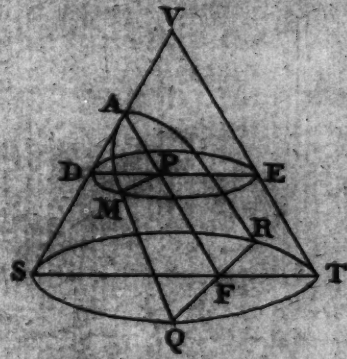
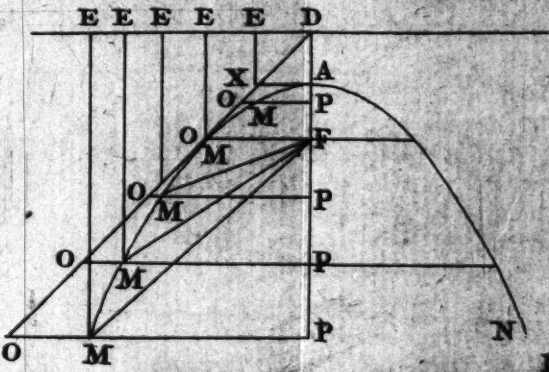
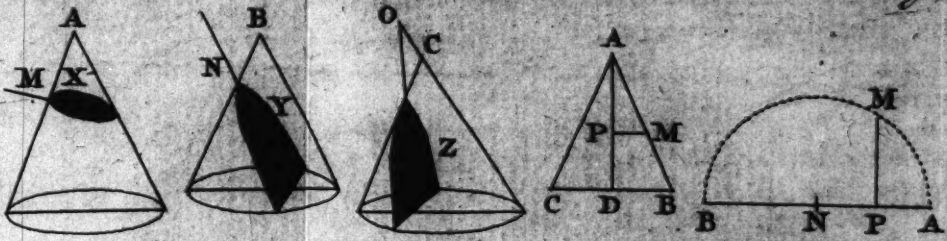
THIS is the Way that Gardeners describe an Ellipsis, 'tis therefore also called the Gardeners Oval, and when the two Diameters are given intersecting at right Angles, to find the Focii H and I, from P or G the Ends of the shortest Diameter, as Centers, and with the Extent of AL or AK, describe part of a Circle crossing LK in H and I which are the said Focii; as is evidently the Problem.

PROB. 4. An Ellipsis being given, to draw from any one of its Points whatsoever as M a Tangent MT.

FROM the Point M let there be drawn to the Focii, F and f, the two Lines MF, Mf; and let fM be prolonged to E, so that ME=MF. Let there be drawn the Line EF. If the Line EF is divided into two equal Parts by the Line MT, I say that MT is a Tangent. Upon this Tangent let there be taken another Point D, from whence draw the Lines DE and DF which are equal, seeing that MT is Perpendicular upon EF. From the Point D also draw the Line Df. Now fD+DE is longer than fM

* Theorem 1.





$fM + ME$; consequently longer than $fM + MF$: For by construction $MF = ME$; therefore the Point D cannot be in the Ellipsis, which hath by the preceeding Theorem this property, that if D be one of its Points, $fD + DF = fM + MF$. The Line DT cannot therefore touch the Ellipsis, but at the Point M .

COROLLARY. It is evident that if AMB represent the Section of an Elliptical Mirrour, and fM be an incident Ray, it will be reflected at F .

For the Angle of Incidence fMS is equal to the Angle of Reflexion FMT ; since that the Angle SMf is equal to EMT , which is equal to TMF . Hence all the Rays which diverge from f , and which fall upon the concave Surface of the Mirrour, will reunite at the Point F , and reciprocally if they diverge from the Point F , they will reunite at the Point f : For this reason it is that these Points are called the *Focii*.

C H A P. IV.

OF the Hyperbola, or Line which represents the Section of a Cone cut by a Plane parallel to its A is, or so that cutting only one Side of the said Cone, it might also cut the other if prolonged above its Vertex.

DEF. 1. LET DE be a Line taken for the Directrix, and F a Point out of that Line. The Line FD is Perpendicular upon DE . If FD is divided at the Point A , according to the Ratio of p to q . (it is here supposed that p is lesser than q) and that a Curve is traced, from which having drawn Perpendiculars to DE as EM , and other Lines to F as FM , such that EM may be to MF as p to q , this Line is called an Hyperbola; because as is demonstrable it hath the same Properties as the Conic Section of that name.

PROB. 1. To find all the Points thro' which this Curve passes.

DRAW AX Parallel to DE , and equal to FA , and draw an indefinite Line on each Side thereof thro' the Points D and X . After which drawing freely as many Parallels to DE as you please, as PO : to find the Point of this Line PO thro' which the Curve passes, describe a Circle from F as a Center, with the Extent FO . The Point M , where the Circle cuts PO , is the Point required; for the Triangles DAX and DPO being similar, $PO : PD :: AX : AD :: q : p$. Now $PO = FM$ by construction. And $PD = EM$: Then $FM : EM :: q : p$. The Point M is therefore in the Hyperbola, by the preceeding Definition.

PROB. 2. To find the Vertex of a Similar, opposite Curve.

In considering the Properties of this Curve, it may be compared with another opposite similar Curve whose Vertex is found thus. Upon FD make the Angle DFx of 45 Degrees, and from

from the Point x where Fx cuts the indefinite Line XD , draw a Perpendicular upon Ff . The Point a upon which this Perpendicular falls, is one of the Points of the Curve: For Fax being Isocles, $ax = aF$. The Triangles $x D a$ and ADX are Similar, by construction; then $a D. ax$ or $a F::DA. AX$ or AF . Hence $a D. a F::DA. AF::p. q$: Then $a D. a F::p. q$. Hence the Point a is one of the Points of an opposite and similar Curve. Having taken $fa = FA$, it gives the Focus f , according to the following Definition.

DEF. 2. THE Line Aa is called the Tranverse or prolonged Axis. The Point C which divides the transverse Axis in two, is called the Center. If thro' C be drawn a Perpendicular BB , whose length is bounded by a Circle described from the Point A , and with the Extent CF , that Line BB is called the conjugate Axis. The Points F and f are the Focii. The Perpendiculars drawn from one of the Points of the Curve upon one of the Axes, are Ordinates. The Abscissa's are the Parts of the Axes taken from their beginning to the meeting of the Ordinates. Sometimes their Origin is at the Center, and sometimes at the Vertex of the Hyperbola's. The third Proportional to the two Axis is called the Parameter of the first Term of the Proportion.

THEOR. 1. THE transverse Axis Aa is to Ff , the Distance of the Focii, as p to q .

Dax and DAX are two similar Triangles: (fig. preced.) then $AX. ax::AD. Da$. Now $AX = AF$, and $ax = aF$: Then $AF. aF::AD. Da$, then compoundedly $AD + Da. AD::AF + Fa. AF$. But $AF + Fa = Ff$: For $AF = af$, and $AD + Da = Aa$. Then $Aa. AD::Ff. AF$, and $Aa. Ff::AD. AF$. Now $AD. AF$, or $AX::p. q$; therefore $Aa. Ff::p. q$; which was to be proved.

THEOR. 2. CD is a third Proportional to CA and CE . (preced fig.)

SEEING that the Wholes are to the Wholes, as the Halves are to the Halves, as CF is the Half of Ff , and CA the half of Aa : Then $Ff. Aa::CF. CA::AX. AD$: Then $CF. CA::CF - AX, or AF. Ca - AD$. Now $CF - AF = CA$, and $Ca - AD = CD$. Therefore $CF. CA::CA. CD$; which was to be proved.

THEOR. 3. THE Square of any Ordinate whatsoever as PM of the transverse Axis Aa , is to the Rectangle of $AP \times pa$ the Parts of that Axis, as the Rectangle $AF \times Fa$ is to the Square of CA or Ca .

LET $CA = d$ the Ordinate $PM = y$, $CP = r$, and $CF = g$; (fig. preced.) when P is above or below the Point F , $PF = x - g$, or $g - x$. Then it will be $DP = \frac{dd}{g}$; for by the preceeding Theo-

rem $CD = \frac{dd}{g}$. But by construction and by the first Theorem,

$d. g::x-\frac{dd}{g}. MF$; then $MF=\frac{gx}{d}-d$. If the Point P were above the Point a , then $MF=\frac{gx}{d}+d$. It must then be proved

that yy or PM . $xx-dd$, or $AP \times Pa::gg-dd$, or $AF \times Fa$. dd , or CA , or Ca ; which is easy: For upon account of the right angled Triangle MPF, $PM^2=FM^2-PF^2$, that is, in Analytical Terms, $yy=\frac{ggxx}{dd}-xx-dd$; or by multiplying both Sides by dd ; it will be $yydd=ggxx-ddxx-ggdd+d^2$, and reducing the Terms composing this Equation into a Proportion, it is $yy. xx-dd::gg-dd. dd$; Q. E. D.

THEOR. 4. THE Square of any Ordinate whatsoever as PM of the Axis Aa is to the Rectangle of the Parts of that Axis (viz. to $AP \times Pa$), as the Square of the conjugate Axis BB is to the Square of the other Axis Aa . (same Figure.)

LET us conceive a right angled Triangle ABC ; according to the Definition of the conjugate Axis, $AB=FC$, it has been supposed $FC=g$, and $AC=d$; let us suppose $BC=b$. Then ABC being right angled, and $AB=g$; then $bb=gg-dd$. Now by the preceeding Theorem, $yy. xx-dd::gg-dd$, or $bb. dd$. Multiplying bb and dd by 4, they remain in the same Ratio. Then $yy. xx-dd::4bb. 4dd$. Now $AP \times Pa=xx-dd$. Therefore the Square of yy is to the Rectangle $AP \times Pa$, as $4bb$, the Square of the conjugate Axis BB or $2b$, is to the Square $4dd$ of the other Axis Aa equal to $2d$.

THEOR. 5. THE Parameter is to the Diameter as the Square of any Ordinate whatsoever, is to the Rectangle of the Parts of that Axis made by the meeting of that Ordinate.

LET the same Denomination of the Parts be preserved and the same Figure as before, and let m be the parameter of the Axis Aa , which is by its definition a third Proportional to the said Axes; hence $\frac{m}{2d}::m. 2b. 2d$. But by reason of this Proportion it will be $m. 2d::4bb. 4dd$. Now by the preceeding Theorem, $4bb$ is to $4dd$ as yy the Square of the Ordinate is to $xx-dd$ the Rectangle of the Parts of the Axis; therefore $m. 2d::yy. xx-dd$, Q. E. D.

THEOR. 6. THE Square of the Ordinates are one to another as the Rectangles of the Parts of the Axis, made by the meeting of the same Ordinates.

WHICH is evident, since that in what place soever the Point P is found, the Parameter m is to the Diameter $2d$, as the Square yy of the given Ordinate is to the Rectangle $xx-dd$, (or $AP \times Pa$) made of the Parts of the Axis limited by the Ordinate.

Theorem

THEOR. 7. THE Hyperbola is the same Curve or Conic Section as that which has been described.

LET mOn be a Cone cut by a Plane, which also meets the other Side nO , of the Cone in B, if prolonged above the Vertex O. This Section qAR is the Curve that has been treated of, or this Curve is really that Section of the Cone which is called *Hyperbola*. Let a Circle DME be conceived Parallel to mqn , that of the Base. Their Diameters cut the Axis of the Hyperbola at the Points P and F. The Lines MP and Fq are Perpendicular upon those Diameters as well as upon the Axis AF, forasmuch as they are the common Sections of two Planes perpendicular to that of the Triangle; consequently they are as much

Ordinates of the Circle, as of the Curve DME. Now $MP^2 =$

$DP \times PE$, and $Fq^2 = mF \times nF$. It is to be demonstrated that $MP^2,$

or $DP \times PE$ is to Fq , or its equal $mF \times nF$, as $AP \times PB$ is to $AF \times FB$. The two Triangles DAP and mAF are Similar. The two Triangles FBn , and PBE are Similar. Therefore $mF.DP :: FA.PA$. Also $nF.PE :: FB.PB$. Each Term of the first Proportion multiplied by its corresponding Term in the Second, the Products are in Proportion, they being Rectangles whose Ratios are compounded of similar Sides. $DP \times PE. mF \times nF :: AP \times PB.$

$AF \times FB$. Then seeing that $MP^2 = DP \times PE$, and $Fq^2 = mF \times nF$.

$MP. Fq :: AP \times PB. AF \times FB$; which was to be proved.

SCHOLIUM,

If the Side of the Cone be prolonged, so that there be supposed another opposite Cone thro' the Vertex thereof, it is plain, that if the intersecting Plane be produced, it will form in the other Cone a similar Section, which is an opposite Hyperbola.

PROB. 3. To describe an Hyperbola by a continued motion.

LET AMX be an Hyperbola, its Vertex is A, and its Focus F. The Vertex of its opposite Hyperbola is B, whose Focus is f . This Hyperbola may be conceived as already described, or it may be described thus. The Line Cf cuts the Hyperbola AMX at the Point M. Let us consider this Line Cf, as a Ruler at the End of which is fastened a Cord whose other Extremity is fixed to the Focus F. This Ruler is fastened by its other End to the Focus of the opposite Hyperbola, and may be turned about its Point f . Let us conceive this Ruler at first, as laid upon AB, and the Cord of such a length that M may then coincide with A. Put a Finger at the Point A or M, and permit it to go towards X, by turning about the Ruler; but always keeping the Chord extended and as fixed against the Ruler. Whilst the

the Ruler turns about, the Finger considered as a Point, will describe the Hyperbola AMX.

COROL. 1. FROM this Construction it follows, that having drawn from each Point of the Hyperbola, two right Lines to the two Focci F and f , the Difference between those two Lines will be always the same.

In its first Position where M is upon A , the Ruler upon AB or upon Ff , it is evident that the Difference between Mf and MF , that is, $Mf - MF$, is AB , since that $FA = fB$; this difference is always the same in the other Positions. For while the Ruler turns about, and as the Finger moves, the Cord MF and the Part Mf of the Ruler encrease their length equally. Therefore these two Quantities preserve the same Difference.

COROL. 2. AN Hyperbola may be extended or continued, ad inf.

FOR if the Ruler Cf be prolonged ad inf. and at the same Time the Cord MC , making the Ruler turn, and continuing to press the Cord as above, it will prolong the Hyperbola without end. It is evident that the Point M will diverge more and more from the Focus F , whilst the Ruler fC is turning about.

SCHOLIUM.

THE Hyperbola being prolonged and continued ad inf. as aforesaid, it will incline more and more to a certain right Line, without ever meeting it, which right Line is called the Asymptote of the Hyperbola; which is a Greek Word, signifying that property.

PROB. 4. To draw the Tangent MT , from any Point whatsoever of an Hyperbola.

LET there be two opposite Hyperbola's A and B . From the given Point M draw the Lines MF , Mf , to the Focci F and f , and divide the Angle FMf into two equal parts by the Line MT ; I say that it is a tangent Line. Let ME be taken equal to MF . Also from any Point whatsoever as D in the Tangent MT , let there be drawn the Lines DF , Df , DE . The Triangles MFN and MEN are equal; as also NDF and NDE . It is required to prove that the Point D which is in the Tangent MT , is not in the Hyperbola; and that therefore this Tangent does neither touch, nor meet in D . If D were in the Hyperbola, $Df - DF = Mf - MF$ by the preceeding Corollary. Now 'tis not so; for ME being taken equal to MF , then $Mf - MF = Ef$. And since that DE is equal to DF , then $Df = Ef + DF$. But in the Triangle DEf , the Sides DE and Ef taken together, are longer than the Side Df ; therefore DE exceeds DF ; then the Point D is not one of those of the Hyperbola; which may be said of every other Point of the Line MT , taken either above or below M ; therefore this Line MT can meet the Hyperbola whose Tangent it is, only in one Point M .

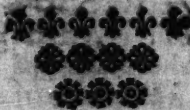
COROLLARY. IF the Curve AM represent the Section of an Hyperbolical Mirrour, and that CM be an incident Ray which falls
upon

upon the concave Surface of the Mirror at the Point *M*, it will therefore reflect from *C* to *F*, and go towards *f*, the Focus of the opposite Hyperbola.

THE Angle *DMC* is equal to the Angle *EMN*, by construction equal to *NMF*; hence *DMC* is equal to *NMF*. The Angle of incidence *CMD* being then equal to the Angle of reflexion *FMT*, the Ray *CM* will be reflected in *F* to the Focus of the Hyperbola. Which ought to be understood of every other incident Ray: and for this reason it is that the Points *F* and *f*, are called Focii.

THESE Curves have innumerable other Properties, which may be seen in an excellent Treatise of Conic Sections, of the late Marquis de L'Hospital.

THE English reader may have Recourse to the Translation of this Work, by EDMUND STONE, F. R. S.

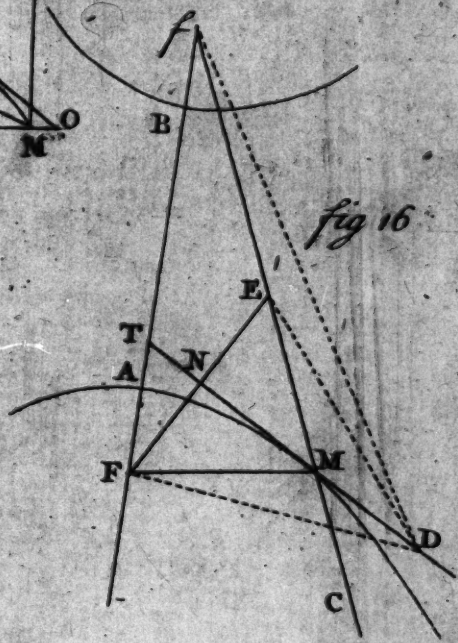
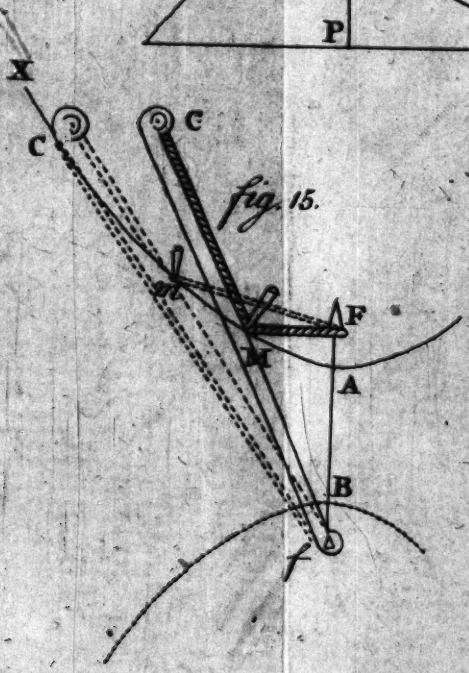
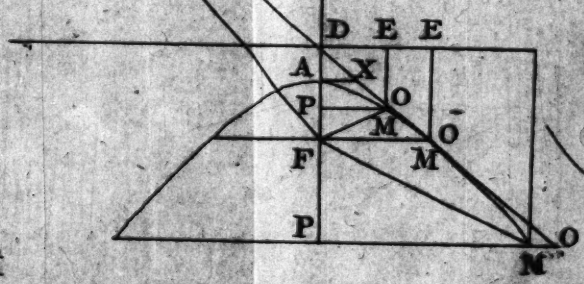
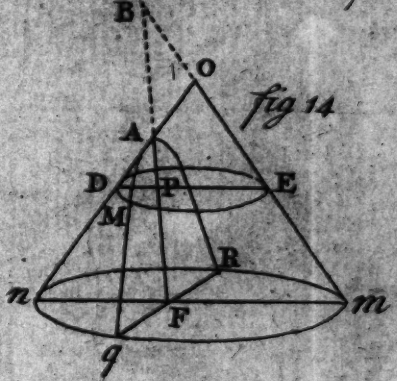
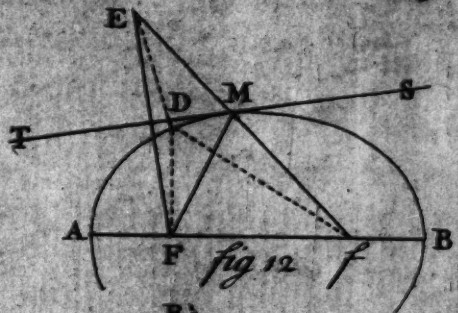
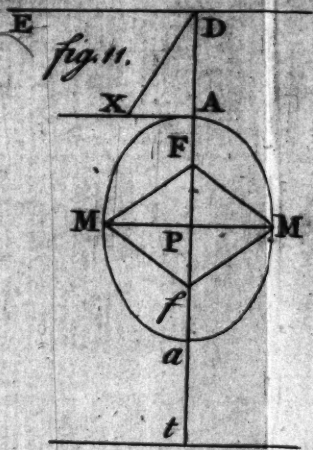


E I N I S.

E

H

C



Exoctaedron or Canted Cube.

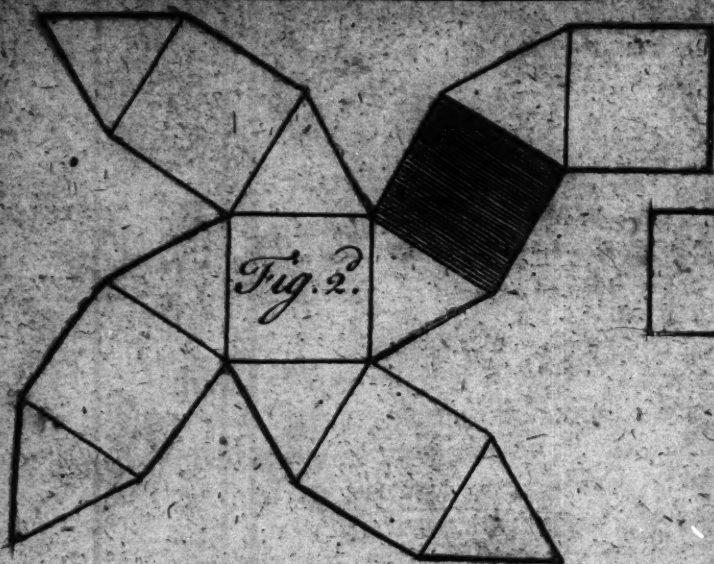


Plate. 1.

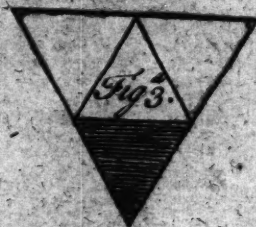
Hexaedron or Cube



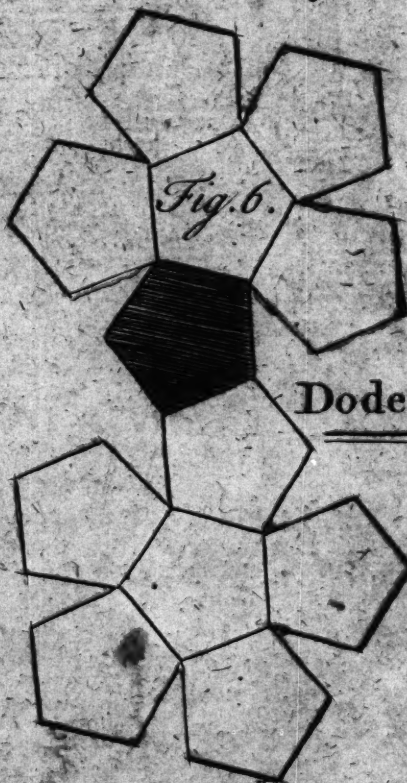
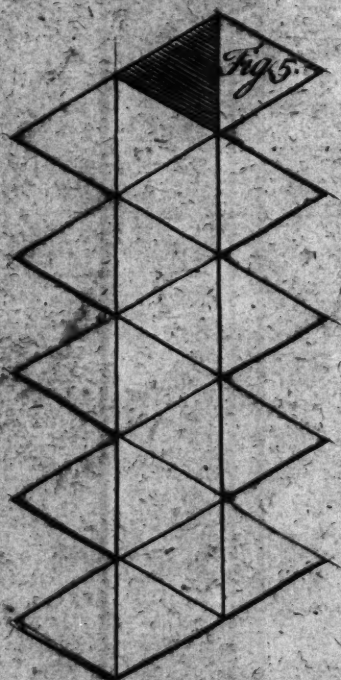
Octoedron.



Tetraedron.



Icofaedron.

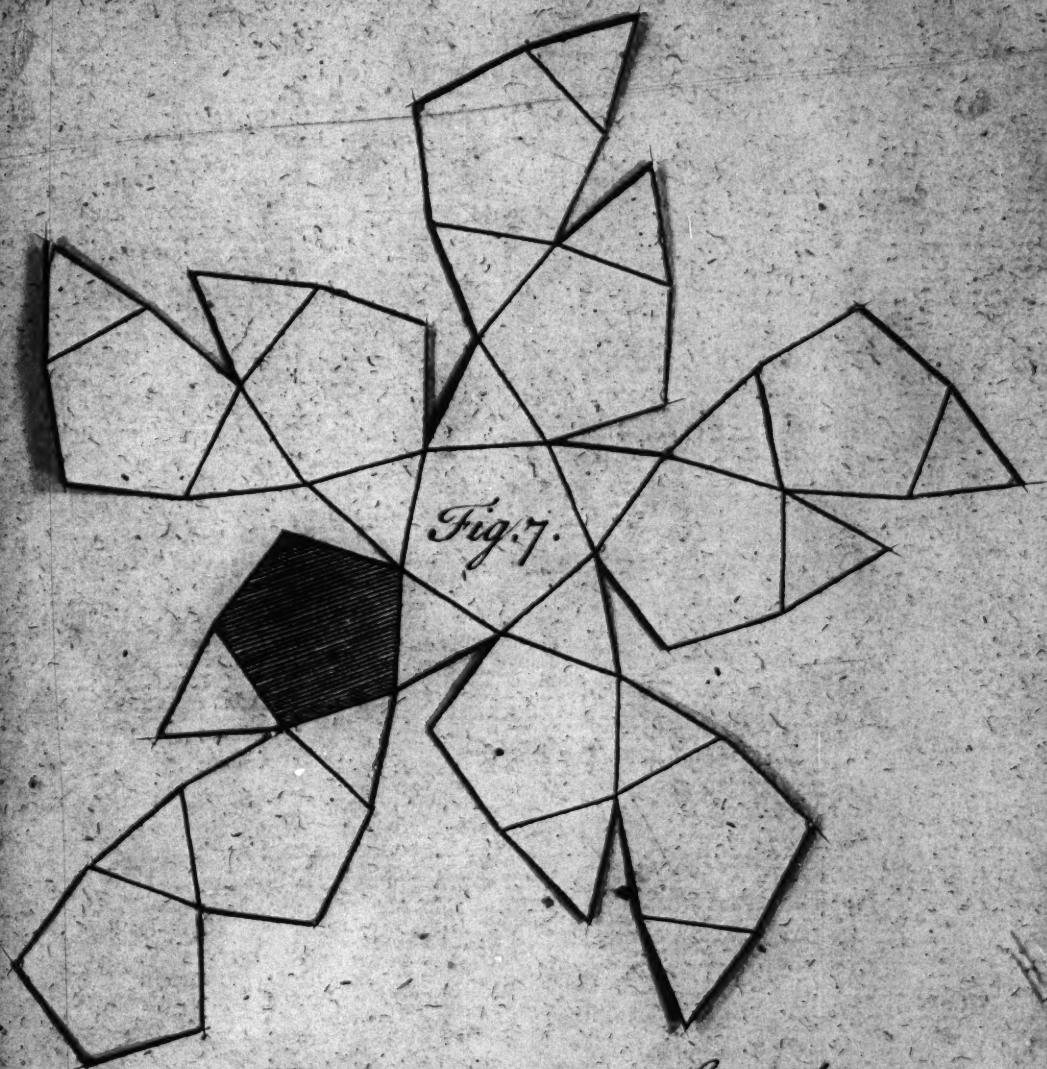


Dodecaedron.

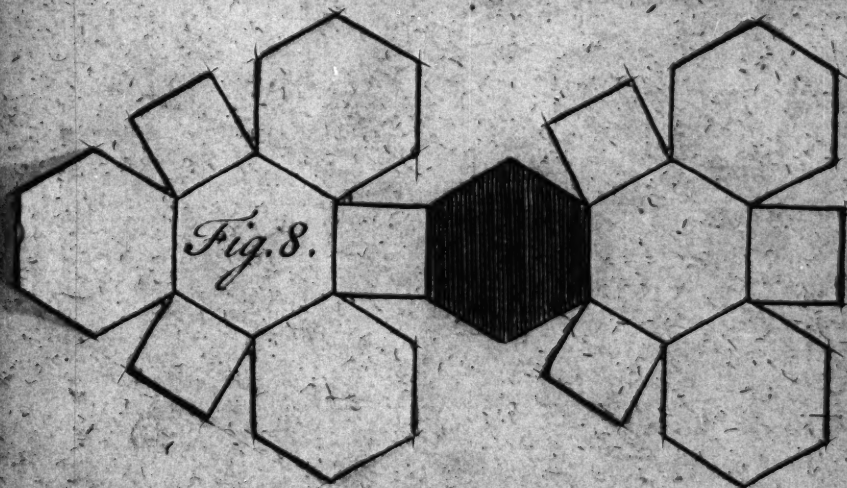
ADVERTISEMENT.

THE Method of folding together the Schemes contained in the following Plates, is so obvious that I apprehend but very little need be said concerning it; shall therefore only observe, that if each of their Sides or Faces be first raised up and pressed down flat, with the Finger, according to the Direction of the Lines which are cut half thro' on the back side, it will occasion the Solid to form more readily. The Part which is shaded is the Base, or part round which the rest are to be folded; except *fig. 15, Plate 5*, where the shade is misplaced, as appears by cutting of that Figure. As to those Figures in *Plate 9* and *10*. which have some of their Lines half cut thro' on one side, and some on the other, it will be proper to begin with folding those Parts first which are nearest the Base, or part which remains fixed, observing when you come to a black Line which is cut half thro' to fold it back, so that the two Sides may lie close together, by which means the several Solids which the whole is divided into, will closely adhere to each other, each appearing distinctly, and which altogether will compose the whole Solid itself, according to its real form before it was so divided.

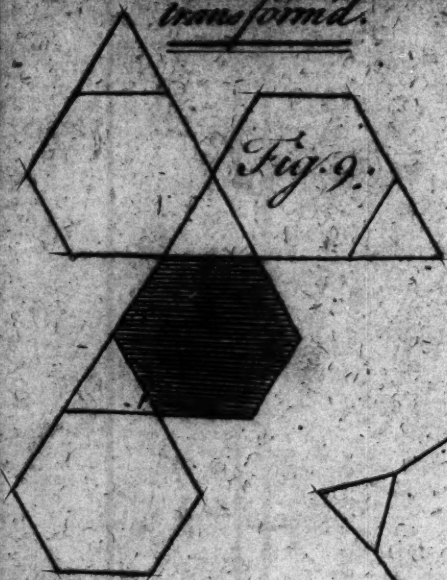
Shortly will be published an additional Plate consisting of two Sides, each of the same size with those contained in this Work, containing the real Inscription of the five regular Solids, treated of in Euclid's fifteenth Book, each body being properly cut, from Schemes which are so adapted that they plainly exhibit the actual Inscription, and Circumscription of each. Price One Shilling.



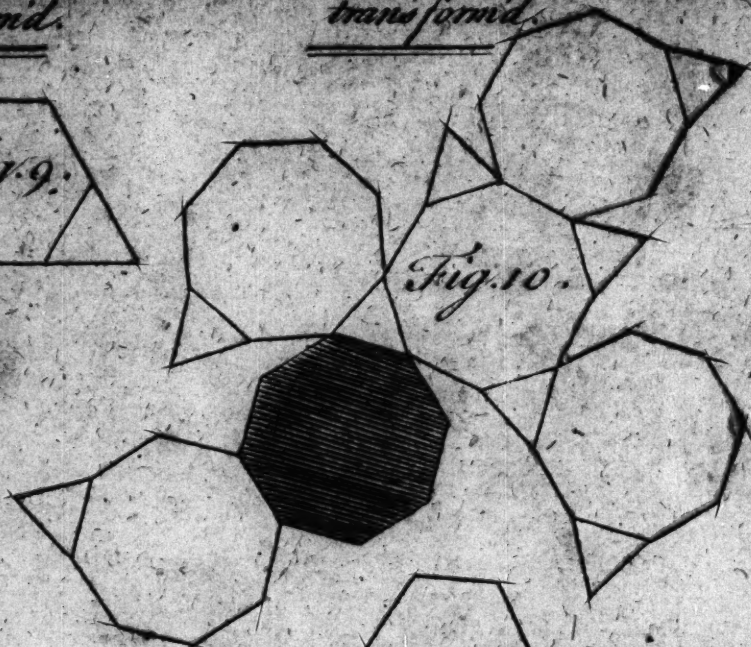
Hexaedron, transformid.



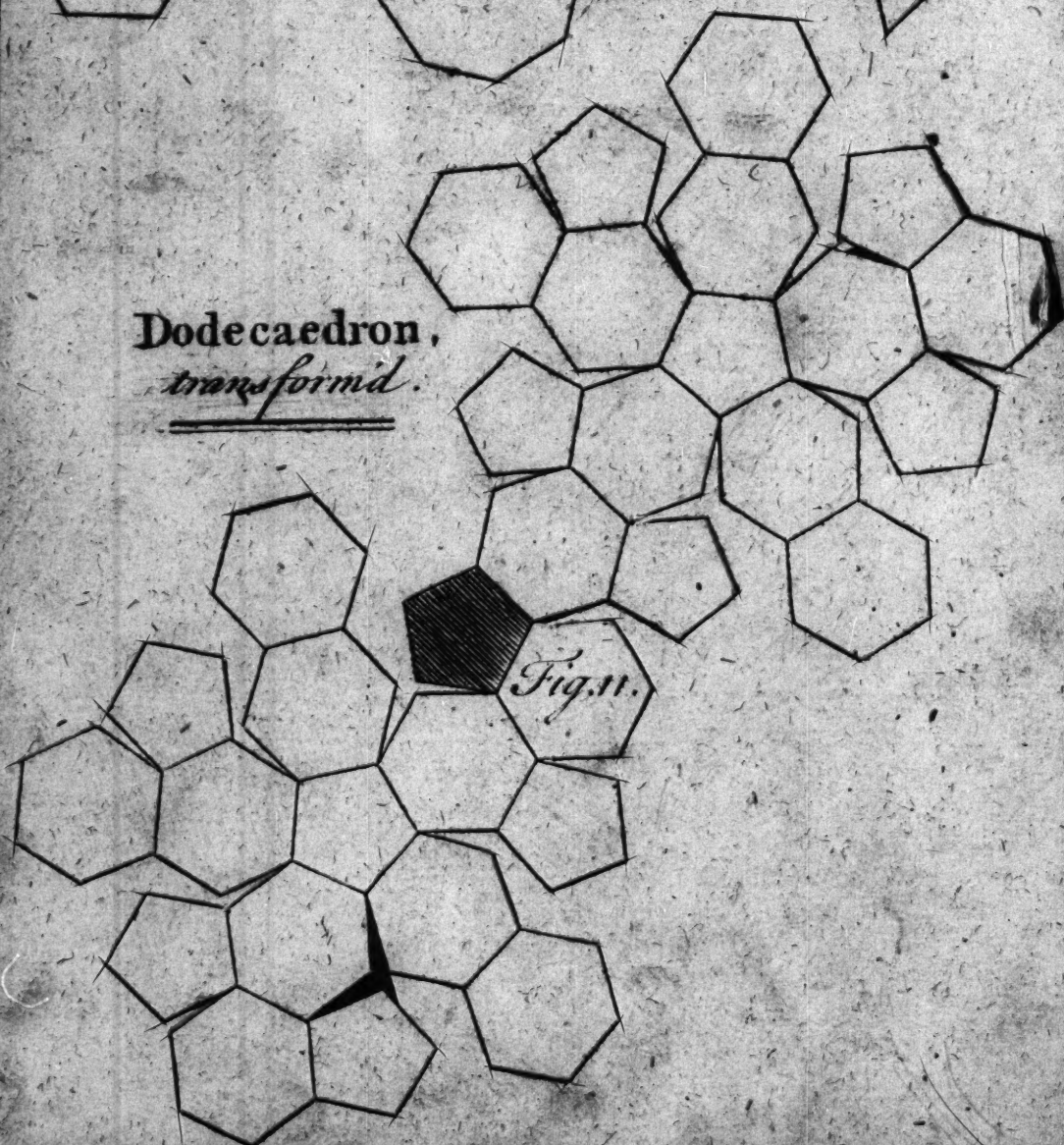
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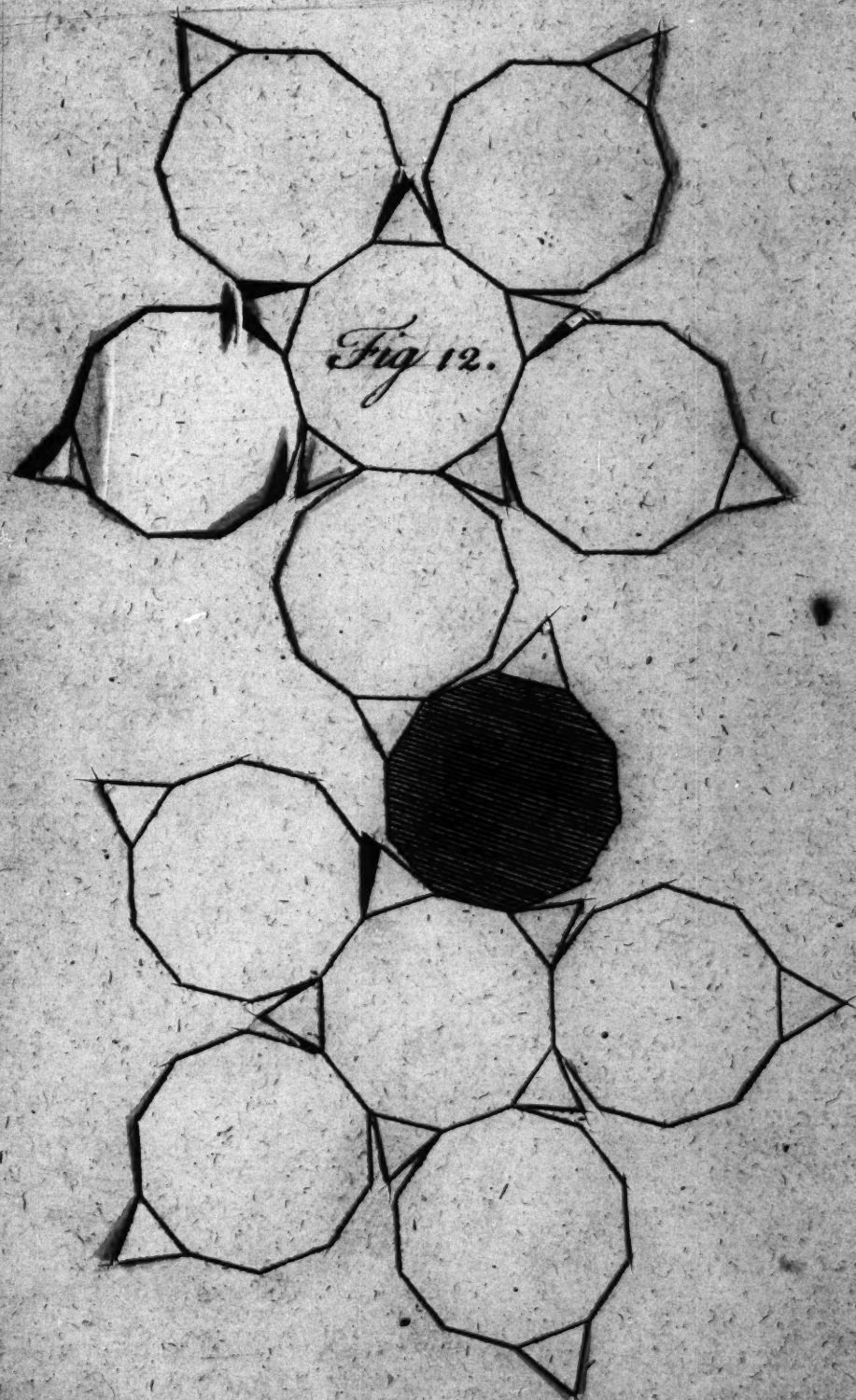
Octoedron, *Plate 3.*
transformid.

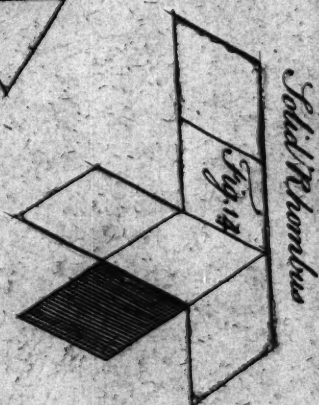
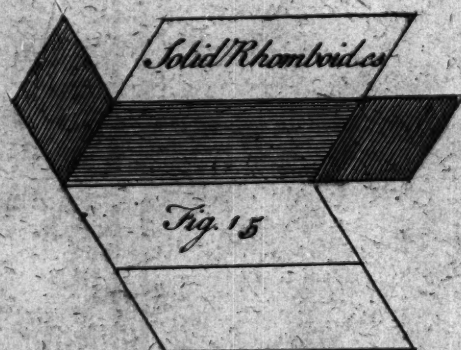
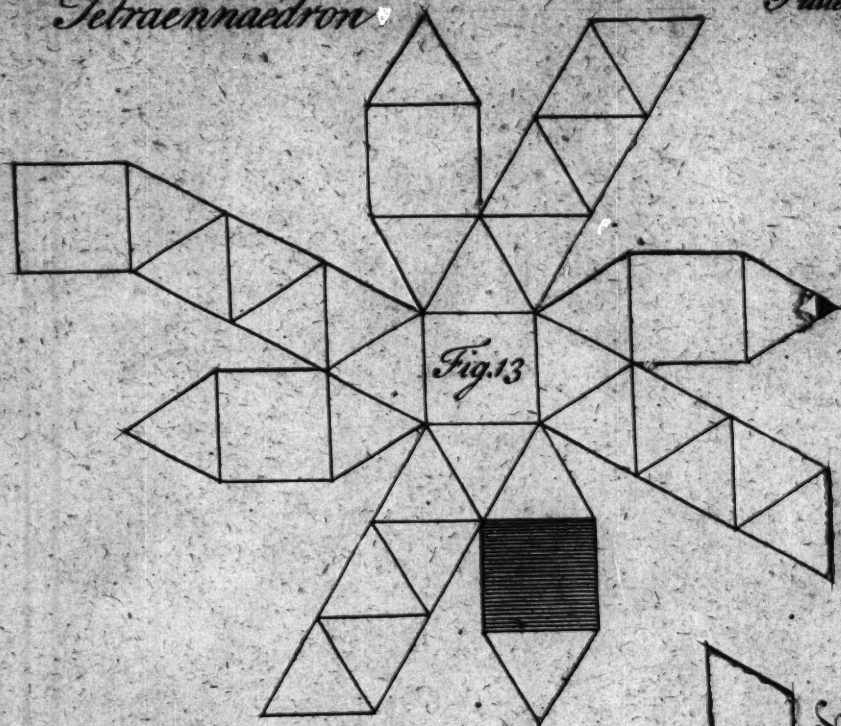


Dodecaedron,
transformid.

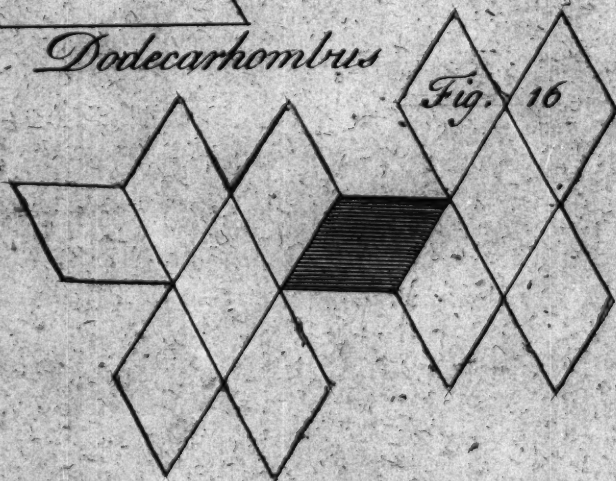


Icosaedron,
transformid.

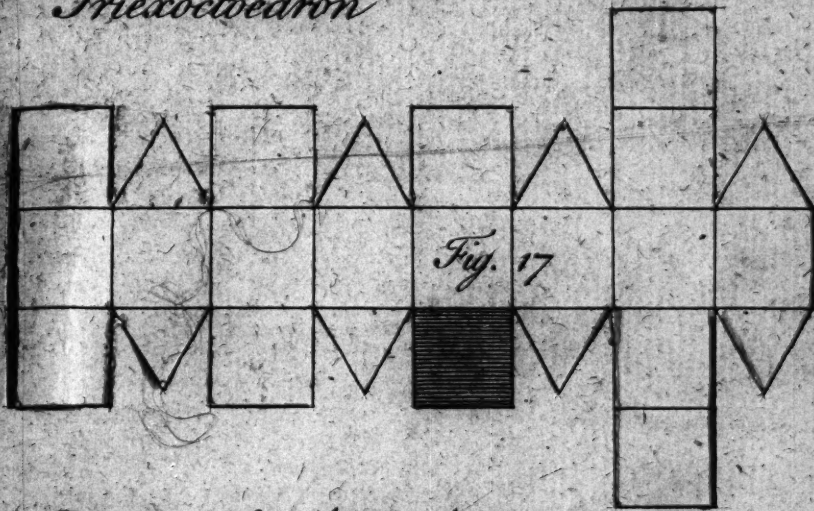




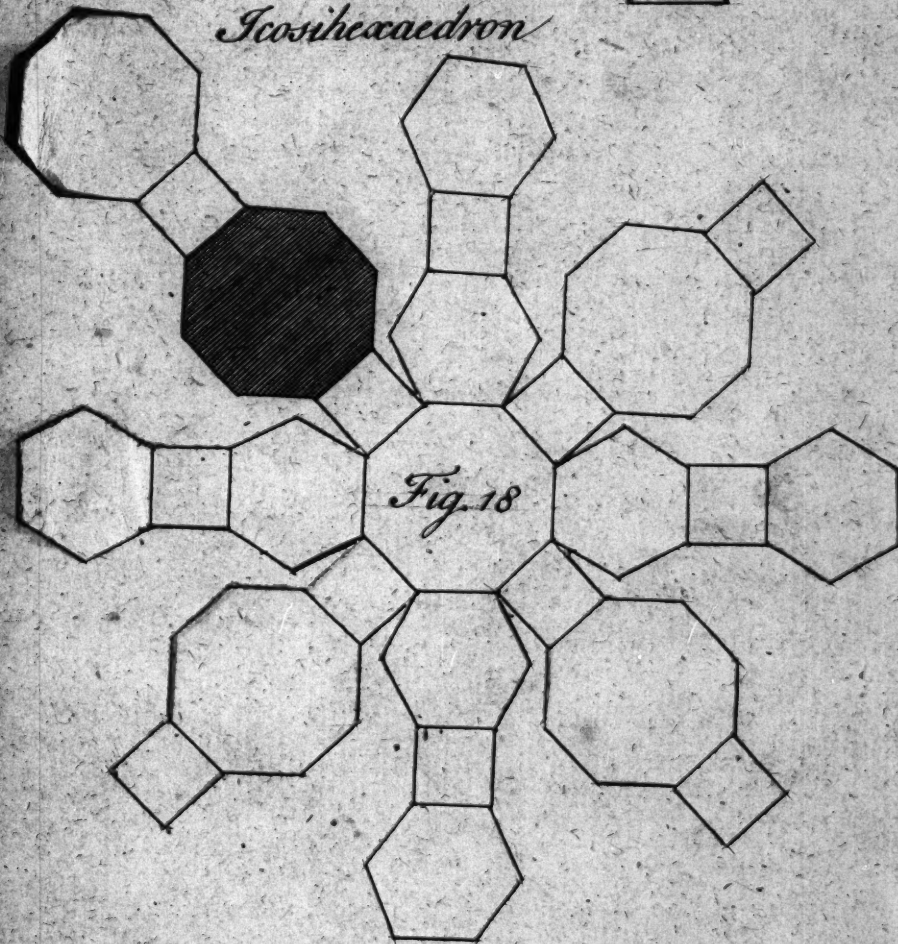
Dodecarrhombrus



Triaoctaedron



Icosihecaedron



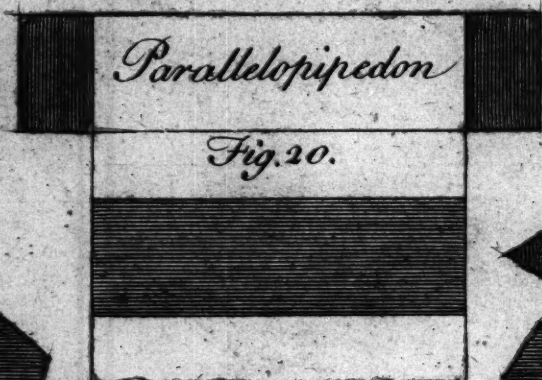


Fig. 20.



Triangular Prism
Fig. 19.

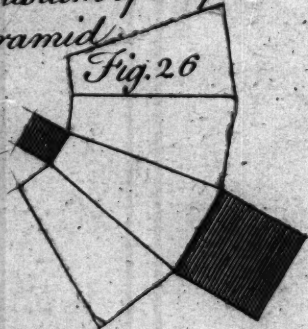


Pentagonal Prism
Fig. 21.

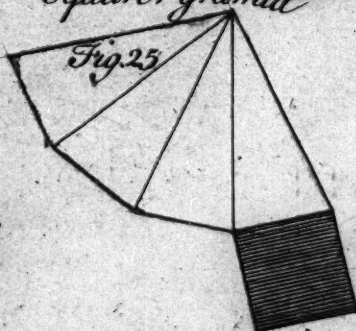


Hexagonal Prism
Fig. 22.

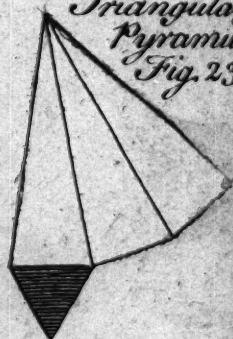
Frustum of a Square Pyramid
Fig. 26



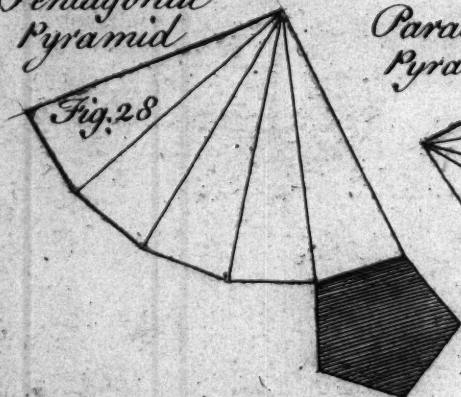
Square Pyramid
Fig. 25



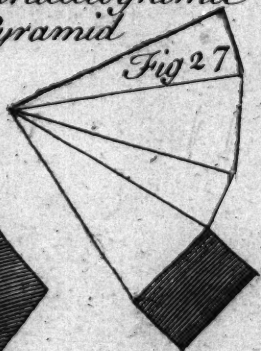
Triangular Pyramid
Fig. 23



Pentagonal Pyramid
Fig. 28



Parallelogramic Pyramid
Fig. 27



Frustum of a Triangular Pyramid
Fig. 24.



A Cone cut obliquely Fig. 30.

Frustum of a
Cone, or a Cone cut parallel
to its Base.
Fig. 29.

A Cone cut
thro' its Axis.

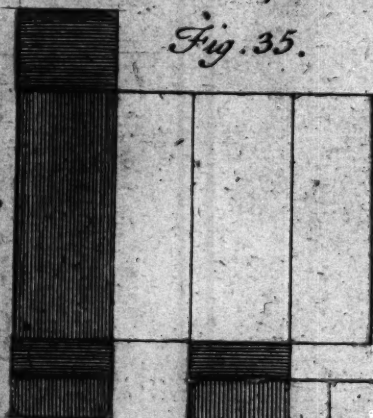
perpendicularly
Fig. 31

A Cone cut
parallel to its Axis
Fig. 33.

A Cone cut
parallel to one
of its Sides.
Fig. 32.

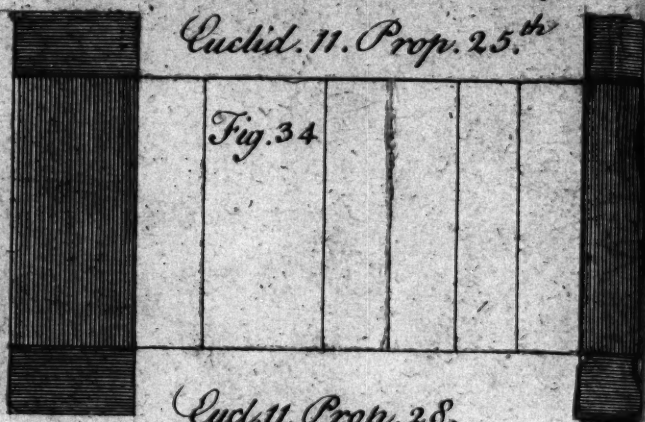
Euclid. 11. Prop. 25.th

Fig. 35.



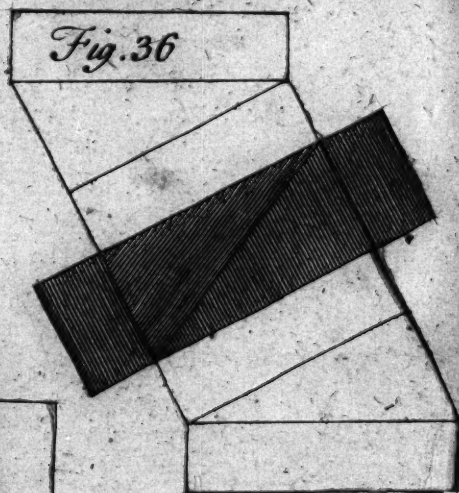
Euclid. 11. Prop. 25.th

Fig. 34



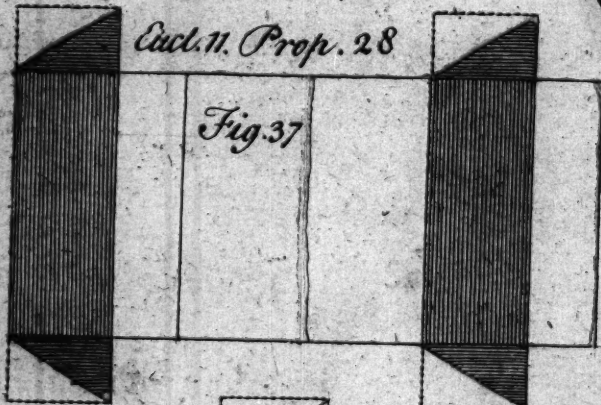
Eucl. 11. Prop. 28.

Fig. 36



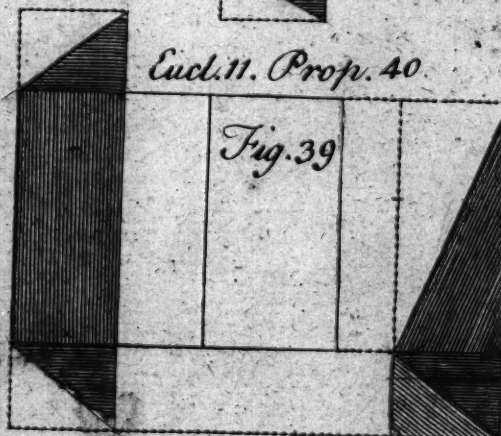
Eucl. 11. Prop. 28

Fig. 37



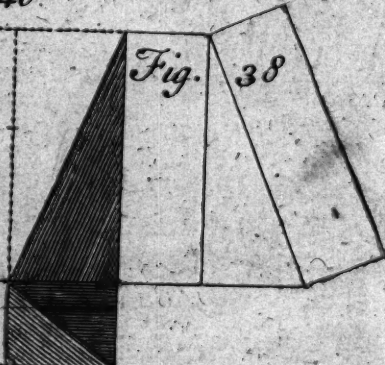
Eucl. 11. Prop. 40

Fig. 39

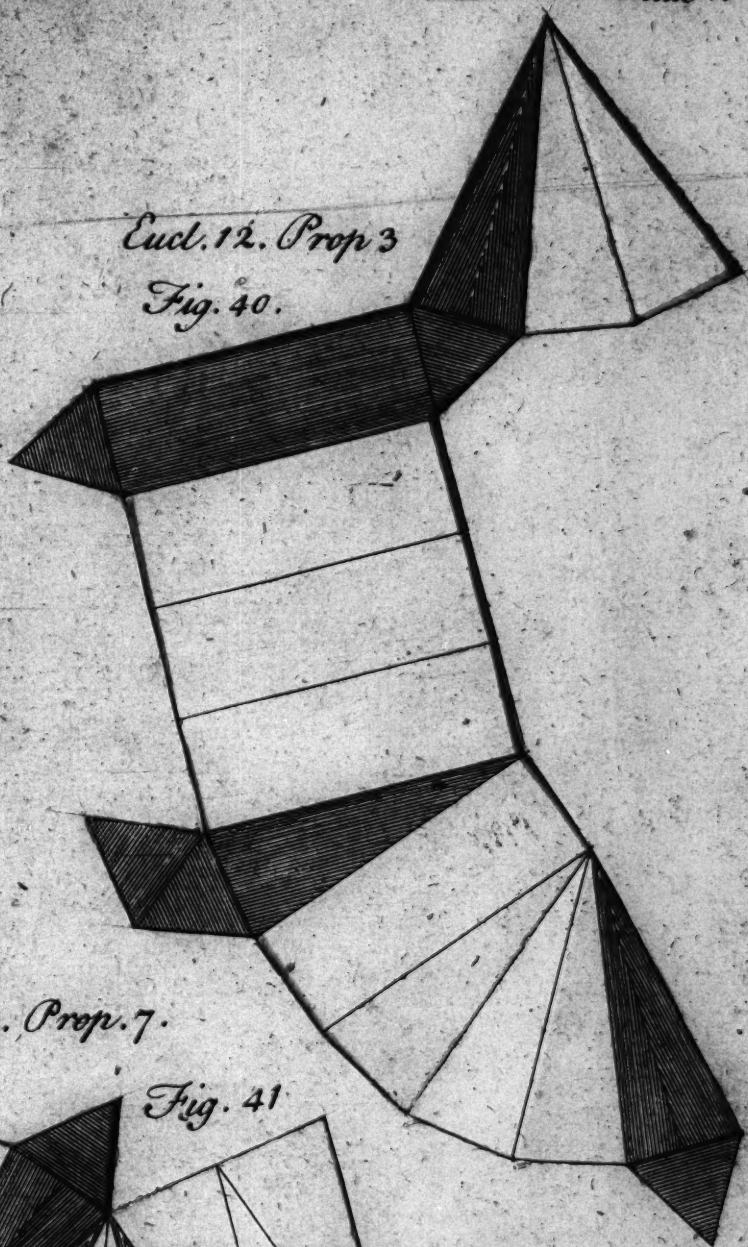


Eucl. 11. Prop. 40

Fig. 38



Eucl. 12. Prop 3
Fig. 40.



Eucl. 12. Prop. 7.

Fig. 41

